

An Extension of LaSalle's Invariance Principle for Switched Systems

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Abstract

This paper addresses invariance principles for a certain class of switched nonlinear systems. We provide an extension of LaSalle's Invariance Principle for these systems and state asymptotic stability criteria. We also present some related results that deal with the compactness of the trajectories of these switched systems and that are interesting by their own.

1 Introduction

In recent years, switched systems have deserved a great deal of attention from the Systems Engineering and Computer Sciences communities. In particular, the stability properties of the common equilibrium solutions have been intensively investigated, see e.g. [3], [9] and [10], respectively, and the references therein. Although switched systems whose component subsystems are autonomous (the class of switched systems that we will consider here) are in essence nonautonomous systems, and as such were investigated by different authors, (see [7] and references therein), their stability properties can also be studied by means of multiple Lyapunov functions, [1], [2], [5], [6], [13] (this approach is very attractive since it enables us to study their uniform, in the sense of the switching signals considered, stability properties). In this context, several Lyapunov-like results and different invariance principles have been recently proposed. Among the invariance results, in [5] LaSalle's invariance principle is extended to switched linear systems under rather general switching; a LaSalle-like invariance principle for switched nonlinear systems under more restrictive switching than that of [5] is proposed in [1] and by using the small-time norm-observability hypothesis, theorems for switched nonlinear systems in the same line as those in [5] are proved in [6]. A version of LaSalle's invariance principle for deterministic hybrid automata with a finite number of discrete states is presented in [12].

In this paper, we extend LaSalle's invariance principle to switched nonlinear systems under mild restrictions in the class of switchings, since we consider switching signals with a positive average dwell-time (see the definition below). This extended principle enables us to obtain (uniform) asymptotic stability criteria for this class of systems.

The results that we present here enable us not only to extend partially some of those in [6] and improve those in [1], but get a better comprehension of the structure of invariant sets and of the compactness properties of trajectories of this class of switched systems.

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The paper is organized as follows. In section 2 we establish the notation, and present the basic definitions and main results of the paper. In section 3 we exhibit some examples that show the application of our results to systems to which those mentioned above cannot be applied or are of little help. We study the invariance for switched systems and prove our Invariance Principle in section 4. Section 5 is devoted to the proof of one of the main results. In section 6 we present the conclusions and finally in the Appendix we present some results about compactness of the trajectories of the switched systems under study. These results, that are used in some proofs along the paper, are also important by their own.

2 Basic definitions and main results

Throughout, \mathbb{R} , \mathbb{R}_+ , \mathbb{N} and \mathbb{N}_0 denote the sets of real, nonnegative real, natural and nonnegative integer numbers, respectively. We use $|\cdot|$ to denote the Euclidean norm on \mathbb{R}^n . As usual, by a \mathcal{K} -function we mean a function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ that is strictly increasing and continuous, and satisfies $\alpha(0) = 0$, by a \mathcal{K}_∞ -function one that is in addition unbounded, and we let \mathcal{KL} be the class of functions $\mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which are of class \mathcal{K}_∞ on the first argument and decrease to zero on the second argument. Let $\mathcal{C}(\mathbb{R}^n, \mathbb{R}^n)$ denote the set of all the continuous maps from \mathbb{R}^n to \mathbb{R}^n .

Given a family $\mathcal{P} = \{f_\gamma \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n) : \gamma \in \Gamma\}$, where $\Gamma = \{1, \dots, m\}$, we consider the switched system described by

$$\dot{x} = f(x, \sigma) \tag{1}$$

where x takes values in \mathbb{R}^n , $\sigma : \mathbb{R}_+ \rightarrow \Gamma$ is a *switching signal*, i.e., σ is piecewise constant and continuous from the right and $f : \mathbb{R}^n \times \Gamma \rightarrow \mathbb{R}^n$ is defined by $f(\xi, \gamma) = f_\gamma(\xi)$. In what follows we consider that the set Γ is equipped with the discrete metric; in consequence Γ is a compact metric space and f a continuous function.

We will denote by \mathcal{S} the set of all the switching signals. We recall that a piecewise smooth curve $x : \mathcal{I} \rightarrow \mathbb{R}^n$, with $\mathcal{I} = [0, T)$ or $[0, T]$ with $0 < T \leq +\infty$ is a *solution* of (1) corresponding to $\sigma \in \mathcal{S}$ if $\dot{x}(t) = f(x(t), \sigma(t))$ for all $t \in (t_i, t_{i+1}) \cap \mathcal{I}$, where $t_0 = 0$ and t_1, t_2, \dots are the consecutive discontinuities (switching times) of σ .

A pair (x, σ) is a *trajectory* of (1) if $\sigma \in \mathcal{S}$ and x is a solution of (1) corresponding to σ . We say that a trajectory (x, σ) of (1) is *maximal* and denote its domain by $\mathcal{I}_{(x, \sigma)} = [0, t_{(x, \sigma)})$, if x is a maximal solution of (1) corresponding to σ . We observe that, due to standard results on ordinary differential equations, either $t_{(x, \sigma)} = \infty$ or $t_{(x, \sigma)} < \infty$ and x is unbounded. Let \mathcal{T} denote the set of all the maximal trajectories of (1).

Since in many applications, the admissible switching signals, and consequently the admissible trajectories of the switched system, are not completely arbitrary since they are subjected to constraints which may concern their functional nature or be dependent on the states $x(t)$ of the system (see [9, 10]), we will suppose that the class of admissible trajectories of (1) is a sub-family \mathcal{T}' of the whole family of trajectories \mathcal{T} .

In order to take into account some kind of state-dependent constraints on the admissible trajectories, as done in [5] we introduce the following family of trajectories: Given a covering $\chi := \{\chi_\gamma : \gamma \in \Gamma\}$ of \mathbb{R}^n , i.e., $\chi_\gamma \subseteq \mathbb{R}^n$ and $\mathbb{R}^n = \bigcup_{\gamma \in \Gamma} \chi_\gamma$, we define $\mathcal{T}[\chi]$ as the set of all the trajectories $(x, \sigma) \in \mathcal{T}$ which verify the condition: $x(t) \in \chi_{\sigma(t)}$ for all $t \in \mathcal{I}_{(x, \sigma)}$. We say that χ is a closed covering if χ_γ is closed for every $\gamma \in \Gamma$.

In what follows we assume that the following standing hypothesis holds

Assumption 1 *There exists a closed covering χ of \mathbb{R}^n such that $\mathcal{T}' \subseteq \mathcal{T}[\chi]$.*

Since we are mainly interested in the stability analysis of the zero solutions of system (1), we will also suppose the following hypothesis holds

Assumption 2 *$f(0, \gamma) = 0$ for all $\gamma \in \Gamma^*$, where $\Gamma^* = \{\gamma \in \Gamma : 0 \in \chi_\gamma\}$;*

and adopt the following definitions of stability.

Definition 2.1 *We say that a family \mathcal{T}' of maximal trajectories of (1) is*

1. *uniformly stable if there exists a function $\alpha \in \mathcal{K}_\infty$ such that for every $(x, \sigma) \in \mathcal{T}'$,*

$$|x(t)| \leq \alpha(|x(t_0)|) \quad \forall t \geq t_0, \forall t_0 \geq 0.$$

2. *Globally asymptotically stable if it is uniformly stable and, in addition, for every trajectory $(x, \sigma) \in \mathcal{T}'$, $x(t)$ converges to 0 as $t \rightarrow \infty$.*
3. *Globally uniformly asymptotically stable if there exists a function $\beta \in \mathcal{KL}$ such that for every $(x, \sigma) \in \mathcal{T}'$,*

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0) \quad \forall t \geq 0, \forall t_0 \geq 0.$$

Remark 2.1 With the same technique used to prove Proposition 2.5 of [11], it can be shown that the definition of global uniform asymptotic stability given above is equivalent to the following (more classical) one:

A family \mathcal{T}' of maximal trajectories of (1) is globally uniformly asymptotically stable if it is uniformly stable and

- (a) for each $R > 0$ and each $\varepsilon > 0$, there exists $T \geq 0$ such that for all $(x, \sigma) \in \mathcal{T}'$ and all $t_0 \geq 0$

$$|x(t_0)| < R \implies |x(t)| < \varepsilon \quad \forall t \geq t_0 + T.$$

Several Lyapunov-like theorems which involve the use of multiple Lyapunov functions (see, among others, [2], [3]) allow us to establish the stability or asymptotic stability of a family of admissible trajectories. The following one which is based on results given in [2], [13] (see also [5]) is an example of such theorems. It is convenient to introduce here the following

Definition 2.2 *A function $V : \mathbb{R}^n \times \Gamma \rightarrow \mathbb{R}$ is a weak Lyapunov-like function for the family \mathcal{T}' if it is continuously differentiable with respect to the first argument and verifies*

1. *there exist α_1 and α_2 of class \mathcal{K}_∞ so that $\alpha_1(|\xi|) \leq V(\xi, \gamma) \leq \alpha_2(|\xi|)$ for all $(\xi, \gamma) \in \mathbb{R}^n \times \Gamma$ such that $\xi \in \chi_\gamma$;*
2. *$\frac{\partial V}{\partial \xi}(\xi, \gamma)f(\xi, \gamma) \leq 0$, for all $(\xi, \gamma) \in \mathbb{R}^n \times \Gamma$ such that $\xi \in \chi_\gamma$;*

3. for every trajectory $(x, \sigma) \in \mathcal{T}'$ and any pair $t_i < t_j$ of switching times such that $\sigma(t_i) = \sigma(t_j)$, $V(x(t_j), \sigma(t_j)) \leq V(x(t_{i+1}), \sigma(t_i))$.

Theorem 2.1 Suppose there exists a weak Lyapunov-like function V for \mathcal{T}' . Then \mathcal{T}' is uniformly stable.

If, in addition, V verifies

- 2'. there exists a positive-definite function α_3 such that $\frac{\partial V}{\partial \xi}(\xi, \gamma)f(\xi, \gamma) \leq -\alpha_3(|\xi|)$, for all $(\xi, \gamma) \in \mathbb{R}^n \times \Gamma$ such that $\xi \in \chi_\gamma$,

then \mathcal{T}' is globally uniformly asymptotically stable.

As was pointed out above, this work is concerned with invariance principles for switched systems and, in particular, with extensions of LaSalle's invariance principle to this class of systems. In this regard we will show, under suitable hypotheses, that the existence of a weak Lyapunov-like function V for \mathcal{T}' , allows us to obtain conclusions about the asymptotic behavior of a bounded solution x of (1) corresponding to some switching signal σ so that $(x, \sigma) \in \mathcal{T}'$, and, further, to obtain some asymptotic stability criteria.

As was discussed in [5] and also in [1], in order to obtain LaSalle-like asymptotic stability criteria by exploiting the knowledge of a weak Lyapunov-like function V , some form of regularity in the switching signals regarding the distance between consecutive switching times is needed. In this paper we will consider switching signals which have a positive average dwell-time, more precisely,

Definition 2.3 We say that the switching signal σ has an average dwell-time $\tau_D > 0$ and a chatter bound $N_0 \in \mathbb{N}$ if the number of switching times of σ in any open finite interval $(\tau_1, \tau_2) \subset \mathbb{R}_+$ is bounded by $N_0 + (\tau_2 - \tau_1)/\tau_D$. \square

We denote by $\mathcal{S}_a[\tau_D, N_0]$ the set of all the switching signals which have an average dwell-time $\tau_D > 0$ and a chatter bound $N_0 \in \mathbb{N}$ and by $\mathcal{T}_a[\tau_D, N_0]$ the subclass of all the trajectories of (1) corresponding to some $\sigma \in \mathcal{S}_a[\tau_D, N_0]$. Let $\mathcal{S}_a = \bigcup_{\tau_D > 0, N_0 > 0} \mathcal{S}_a[\tau_D, N_0]$ and let \mathcal{T}_a denote the subclass of all the trajectories of (1) corresponding to some $\sigma \in \mathcal{S}_a$, i.e. $\mathcal{T}_a = \bigcup_{\tau_D > 0, N_0 > 0} \mathcal{T}_a[\tau_D, N_0]$.

We note that the set of switching signals σ which have a dwell-time $\tau_D > 0$, i.e., $\inf_{k \geq 0} t_{k+1} - t_k \geq \tau_D$, is a subset of $\mathcal{S}[\tau_D, 1]$.

From now on we suppose that the following additional hypothesis holds.

Assumption 3 $\mathcal{T}' \subseteq \mathcal{T}_a$.

In order to establish the main results of this paper, we need to introduce some more trajectory families.

Given a continuous function $V : \Omega \times \Gamma \rightarrow \mathbb{R}$, with Ω an open subset of \mathbb{R}^n , we consider the following families of trajectories associated with V .

\mathcal{T}_V is the class of all the trajectories $(x, \sigma) \in \mathcal{T}$ which verify the conditions:

1. $x(t) \in \Omega$ for all $t \in \mathcal{I}_{(x, \sigma)}$;
2. for any pair of times $t, t' \in \mathcal{I}_{(x, \sigma)}$ such that $t \leq t'$ and $\sigma(t) = \sigma(t')$, $V(x(t), \sigma(t)) \geq V(x(t'), \sigma(t'))$.

\mathcal{T}_V^* is the sub-family of \mathcal{T}_V whose members (x, σ) verify the condition

$$\sigma(t) = \sigma(t') \implies V(x(t), \sigma(t)) = V(x(t'), \sigma(t')).$$

Remark 2.2 If V is a weak Lyapunov-like function for \mathcal{T}' it readily follows that $\mathcal{T}' \subseteq \mathcal{T}_V$.

Finally, we introduce the following notion of weak-invariance for nonempty subsets of $\mathbb{R}^n \times \Gamma$.

Definition 2.4 Given a family \mathcal{T}^* of maximal trajectories of (1), we say that a nonempty subset $M \subseteq \mathbb{R}^n \times \Gamma$ is weakly-invariant with respect to \mathcal{T}^* if for each $(\xi, \gamma) \in M$ there is a trajectory $(x, \sigma) \in \mathcal{T}^*$ such that $x(0) = \xi$, $\sigma(0) = \gamma$ and $(x(t), \sigma(t)) \in M$ for all $t \in \mathcal{I}_{(x, \sigma)}$. \square

Now we are in position to state the following asymptotic stability criterion, which is one of our main results.

Theorem 2.2 Suppose that there exists a weak Lyapunov-like function V for \mathcal{T}' such that $M = \{0\} \times \Gamma^*$ is the maximal weakly-invariant set w.r.t. $\mathcal{T}_V^* \cap \mathcal{T}[\chi] \cap \mathcal{T}_a$.

Then \mathcal{T}' is globally asymptotically stable. If, in addition, $\mathcal{T}' \subseteq \mathcal{T}_a[\tau_D, N_0]$ for some $\tau_D > 0$ and some $N_0 \in \mathbb{N}$ then \mathcal{T}' is globally uniformly asymptotically stable. \square

Remark 2.3 From the proof of Theorem 2.2, which can be found in section 5, it follows that in the case when $\mathcal{T}' \subseteq \mathcal{T}_a[\tau_D, N_0]$ for some $\tau_D > 0$ and some $N_0 \in \mathbb{N}$, the thesis of the theorem still holds if one assumes the weaker hypothesis $M = \{0\} \times \Gamma^*$ is the maximal weakly-invariant set w.r.t. $\mathcal{T}_V^* \cap \mathcal{T}[\chi] \cap \mathcal{T}_a[\tau_D, N_0]$.

The following result can be readily deduced from Theorem 2.2.

Theorem 2.3 Suppose that there exists a weak Lyapunov-like function V for \mathcal{T}' . Suppose, in addition, that there exists a family $\{W_\gamma : \mathbb{R}^n \rightarrow \mathbb{R}, \gamma \in \Gamma\}$ of continuous and nonnegative definite functions such that

1. $\frac{\partial V}{\partial \xi}(\xi, \gamma)f(\xi, \gamma) \leq -W_\gamma(\xi)$, for all $(\xi, \gamma) \in \mathbb{R}^n \times \Gamma$ such that $\xi \in \chi_\gamma$;
2. for each $\gamma \in \Gamma$, the system

$$\dot{x} = f_\gamma(x), \quad y = W_\gamma(x), \tag{2}$$

is zero small-time distinguishable. (We recall that a systems (2) is zero small-time distinguishable, if for every $\delta > 0$, $x(0) = 0$ whenever $W_\gamma(x(t)) = 0$ for all $t \in [0, \delta)$).

Then \mathcal{T}' is globally asymptotically stable. If, in addition, $\mathcal{T}' \subseteq \mathcal{T}_a[\tau_D, N_0]$ for some $\tau_D > 0$ and some $N_0 \in \mathbb{N}$ then \mathcal{T}' is globally uniformly asymptotically stable.

Proof. Let M be the maximal weakly-invariant set w.r.t. $\mathcal{T}_V^* \cap \mathcal{T}[\chi] \cap \mathcal{T}_a$. In order to prove the theorem, it suffices to show that $M = \{0\} \times \Gamma^*$, since then the hypotheses of Theorem 2.2 will be fulfilled. Note first that $\{0\} \times \Gamma^*$ is weakly-invariant w.r.t. $\mathcal{T}_V^* \cap \mathcal{T}[\chi] \cap \mathcal{T}_a$ due to Assumption 2 and the fact that, due to 1. in Definition 2.2, $V(0, \gamma) = 0 \forall \gamma \in \Gamma^*$.

Let then $(\xi, \gamma) \in M$; it follows that there exists a trajectory $(x, \sigma) \in \mathcal{T}_V^* \cap \mathcal{T}[\chi] \cap \mathcal{T}_a$ such that $(x(0), \sigma(0)) = (\xi, \gamma)$ and $(x(t), \sigma(t)) \in M$ for all $t \geq 0$. Let $\tau > 0$ such that $\sigma(t) = \gamma$ for all $t \in [0, \tau)$. It follows from the definition of \mathcal{T}_V^* that $\frac{\partial V}{\partial \xi}(x(t), \sigma(t))f(x(t), \sigma(t)) = 0$ for all $t \in [0, \tau)$ and hence $W_\gamma(x(t)) = 0$ for every $t \in [0, \tau)$. From the zero small-time distinguishability assumption, we have that $\xi = x(0) = 0$. Since $(x, \sigma) \in \mathcal{T}[\chi]$, $x(0) \in \chi_{\sigma(0)}$ and consequently $\gamma \in \Gamma^*$. ■

Remark 2.4 Theorem 2.3 is a partial generalization of Theorem 7 in [6], since the zero small-time distinguishability hypothesis is weaker than the small-time norm-observability Assumption 2. in [6] and since we obtain *uniform* asymptotic stability in the case when $\mathcal{T}' \subseteq \mathcal{T}_a[\tau_D, N_0]$ for some $\tau_D > 0$ and some $N_0 \in \mathbb{N}_0$, but our hypothesis $\mathcal{T}' \subseteq \mathcal{T}_a$ is slightly stronger than the hypothesis about the regularity of the switching signals considered in [6] (see Assumption 3. of that paper).

The proof of Theorem 2.2 is based on the following extension of the well known invariance principle for dynamical systems described by differential equations of LaSalle (see [8]) to switched systems. This extension is other of the main results of this work.

Let $\pi_1 : \mathbb{R}^n \times \Gamma \rightarrow \mathbb{R}^n$ be the projection onto the first component.

Theorem 2.4 Let χ be a closed covering of \mathbb{R}^n and let $V : \Omega \times \Gamma \rightarrow \mathbb{R}$, with Ω an open subset of \mathbb{R}^n , be continuous. Suppose that (x, σ) is a trajectory belonging to $\mathcal{T}_V \cap \mathcal{T}[\chi] \cap \mathcal{T}_a[\tau_D, N_0]$ such that for some compact subset $B \subset \Omega$, $x(t) \in B$ for all $t \geq 0$. Let $M \subseteq \mathbb{R}^n \times \Gamma$ be the largest weakly-invariant set w.r.t. $\mathcal{T}_V^* \cap \mathcal{T}[\chi] \cap \mathcal{T}_a[\tau_D, N_0]$ contained in $\Omega \times \Gamma$.

Then, $x(t)$ converges to $\pi_1(M)$ as $t \rightarrow \infty$.

Remark 2.5 In the case when we restrict the hypotheses of Theorem 2.4 to those of Theorems 1 and 2 in [1], we obtain more precise results related to the size of the attracting sets involved. We shall prove our assertion for Theorem 1 in [1] only, since the proof for the other is similar. In what follows, and in order to prove our claim, we refer to the notation and definitions of that paper. Let $x(t)$ be a dwell-time solution (in the sense of [1]) with initial condition $x(0) \in \Omega_l$, a dwell-time $\tau_D > 0$ and generated by a switching signal σ in a switched system that admits a common weak Lyapunov function (in the sense above) $V : \Omega \rightarrow \mathbb{R}_+$. Let $W : \Omega_l \rightarrow \mathbb{R}_+$ be the restriction to Ω_l of the function V . It is easy to see that the trajectory (x, σ) belongs to $\mathcal{T}_W \cap \mathcal{T}_a[\tau_D, 1]$, and that there exists a compact set $B \subset \Omega_l$ so that $x(t) \in B$ for all $t \geq 0$. Therefore the trajectory (x, σ) verifies the hypotheses of Theorem 2.4 (with W in place of V and the trivial covering of \mathbb{R}^n , $\chi_\gamma = \mathbb{R}^n$ for all $\gamma \in \Gamma$). Hence $x(t)$ converges to $\pi_1(M)$ as $t \rightarrow \infty$, where M is the largest weakly-invariant set w.r.t. $\mathcal{T}_W^* \cap \mathcal{T}_a$ contained in $\Omega_l \times \Gamma$.

On the other hand, Theorem 1 asserts that $x(t)$ converges to M' , where M' is the union of all the compact, weakly-invariant sets (in the sense of [1]) which are contained in $Z \cap \Omega_l$. We will prove that $\pi_1(M) \subseteq M'$ and therefore our assertion about the sizes of the attracting sets.

Pick $\xi \in \pi_1(M)$. Then there exist $\gamma \in \Gamma$ and a trajectory $(x^*, \sigma^*) \in \mathcal{T}_W^* \cap \mathcal{T}_a$ such that $(x^*(0), \sigma^*(0)) = (\xi, \gamma)$ and $(x^*(t), \sigma^*(t)) \in M$ for all $t \in \mathcal{I}_{(x^*, \sigma^*)}$. In consequence $\sigma^*(t) = \gamma$ for all $t \in [0, \delta]$ with $0 < \delta < t_1$ and t_1 the first switching time of σ^* . It follows from the definition of \mathcal{T}_W^* that $\frac{\partial V}{\partial \xi}(x^*(t))f(x^*(t), \gamma) = 0$ and hence that $x^*(t) \in Z \cap \Omega_l$ for all $t \in [0, \delta]$. Next, $x^*([0, \delta])$ is a compact weakly-invariant set contained in $Z \cap \Omega_l$ and consequently $\xi \in M'$.

It must be remarked that the attracting set $\pi_1(M)$ corresponding to the weakly-invariant set considered in Theorem 2.4, may be considerably smaller than the attracting sets given in Theorems 1

and 2 of [1], as we exhibit in Example 2 below.

3 Examples

Example 1. Consider the switched system in \mathbb{R}^2 given by the family $\{f_1, f_2\}$, with

$$f_1(\xi) = \begin{pmatrix} -2\xi_1 - 2\xi_2 \\ 2\xi_1 \end{pmatrix}, \quad f_2(\xi) = \begin{pmatrix} -\xi_2 \\ \xi_1 \end{pmatrix}.$$

Let \mathcal{T}' be the set of all the maximal trajectories (x, σ) whose switching signals σ are given by the feedback rule

$$\sigma(t) = \begin{cases} 1 & \text{if } x_1(t) < 0 \\ 2 & \text{if } x_1(t) \geq 0. \end{cases}$$

Observe that the origin is a stable focus for the first subsystem and a center for the other, and that the trajectories of both are running counterclockwise.

Since the time needed by any nontrivial trajectory of the subsystem $\dot{x} = f_1(x)$ ($\dot{x} = f_2(x)$) to go from the positive (resp. negative) x_2 axis to the negative (resp. positive) one is constant, clearly $\mathcal{T}' \subseteq \mathcal{T}_a[\tau_D, 1]$ for some $\tau_D > 0$. If we consider the closed covering of \mathbb{R}^2 : $\chi_1 = \{\xi : \xi_1 \leq 0\}$, $\chi_2 = \{\xi : \xi_1 \geq 0\}$, then $\mathcal{T}' \subseteq \mathcal{T}[\chi] \cap \mathcal{T}_a[\tau_D, 1]$.

The function $V : \mathbb{R}^2 \times \{1, 2\} \rightarrow \mathbb{R}$ defined by $V(\xi, i) = |\xi|^2$ is clearly a weak Lyapunov-like function for \mathcal{T}' . We claim that $M = \{0\} \times \{1, 2\}$. In fact, let $(\xi, \gamma) \in M$. Then there exists $(x, \sigma) \in \mathcal{T}_V^* \cap \mathcal{T}[\chi] \cap \mathcal{T}_a$ such that $x(0) = \xi$, $\sigma(0) = \gamma$ and $(x(t), \sigma(t)) \in M$ for all $t \geq 0$. From the facts that $x(t)$ cannot remain forever in the right half-plane, where $V(x(t), \sigma(t))$ is constant, that $V(x(t), \sigma(t))$ is strictly decreasing when $x(t)$ is in the open left-half plane, and from the definition of \mathcal{T}_V^* , it follows readily that $(x(t), \sigma(t))$ cannot belong to M unless $x(t) = 0$ for all $t \geq 0$, and the claim follows. Hence, according to Theorem 2.2, \mathcal{T}' is globally uniformly asymptotically stable.

Example 2. Consider now the two systems in \mathbb{R}^2 given by:

$$\dot{x} = f_1(x) = \begin{pmatrix} -x_1 - x_2 \\ x_1 \end{pmatrix} \quad \text{and} \quad \dot{x} = f_2(x) = -\frac{x}{1 + |x|^4}.$$

Let $W_1(\xi) = \xi_1^2$, $W_2(\xi) = \frac{|\xi|^2}{1 + |\xi|^4}$ and $V(\xi, i) = |\xi|^2/2$, $i = 1, 2$. Then for every $\xi \in \mathbb{R}^2$, $\frac{\partial V}{\partial \xi}(\xi, 1)f_1(\xi) = -W_1(\xi)$ and $\frac{\partial V}{\partial \xi}(\xi, 2)f_2(\xi) = -W_2(\xi)$. It is not hard to see that both pairs (f_1, W_1) and (f_2, W_2) have the zero small-time distinguishability property and that the second one is not small-time norm-observable. As a matter of fact it is not large-time norm observable. In this case Theorem 7 in [6] cannot be applied, but according to Theorem 2.3, any $\mathcal{T}' \subseteq \mathcal{T}_a[\tau_D, N_0]$ with fixed, but otherwise arbitrary, $\tau_D > 0$, $N_0 \in \mathbb{N}_0$, is globally uniformly asymptotically stable.

It is worth noting that if one applied Theorem 1 of [1], the attracting set so obtained would be the x_2 -axis (see example 4 of [1]).

4 Invariance for switched systems

In this section we study the asymptotic behavior of bounded solutions x of the switching system (1) corresponding to switching signals which have a positive average dwell-time, and in particular the

invariance properties of their ω -limit sets. We recall that a point $\xi \in \mathbb{R}^n$ belongs to $\Omega(x)$, the ω -limit set of $x : \mathbb{R}_+ \rightarrow \mathbb{R}^n$, if there exists a strictly increasing sequence of times $\{s_k\}$ with $\lim_{k \rightarrow \infty} s_k = \infty$ and $\lim_{k \rightarrow \infty} x(s_k) = \xi$. The ω -limit set $\Omega(x)$ is always closed and, when x is bounded, it is non-empty, compact and $\lim_{t \rightarrow \infty} d(x(t), \Omega(x)) = 0$. Moreover, $\Omega(x)$ is the smallest closed set which is approached by x .

In order to proceed, we will associate to each bounded trajectory $(x, \sigma) \in \mathcal{T}_a$ a nonempty subset of $\mathbb{R}^n \times \Gamma$, which we denote $\Omega^\sharp(x, \sigma)$, and study its invariance properties.

Let us introduce some more notation and terminology. As stated above, associated with a switching signal σ there are a strictly increasing sequence of real numbers (the sequence of switching times of σ) $\{t_i\}_{i=0}^{N_\sigma}$, with N_σ finite or $N_\sigma = \infty$, $t_0 = 0$ and $\lim_{i \rightarrow \infty} t_i = \infty$ when $N_\sigma = \infty$, and a sequence of points $\{\gamma_i\}_{i=0}^{N_\sigma} \subseteq \Gamma$, with $\gamma_i \neq \gamma_{i+1}$ for all $0 \leq i < N_\sigma$, such that $\sigma(t) = \gamma_i$ for all $t_i \leq t < t_{i+1}$ with $0 \leq i < N_\sigma$, and $\sigma(t) = \gamma_{N_\sigma}$ for all $t \geq t_{N_\sigma}$ when N_σ is finite. In order to treat the cases $N_\sigma < \infty$ and $N_\sigma = \infty$ in an unified frame, we pick any $\gamma^* \in \Gamma$ and define $t_i = \infty$ and $\gamma_i = \gamma^*$ for all $i > N_\sigma$ when N_σ is finite.

Given a switching signal $\sigma \in \mathcal{S}$ we consider the sequence of maps $\tau_\sigma^i : \mathbb{R}_+ \cup \{\infty\} \rightarrow \mathbb{R}_+ \cup \{\infty\}$, $i \in \mathbb{N}$, defined recursively by:

- $\tau_\sigma^1(t) = t_k$ if $t \in [t_{k-1}, t_k)$ and $\tau_\sigma^1(\infty) = \infty$;
- $\tau_\sigma^{i+1}(t) = \tau_\sigma^1(\tau_\sigma^i(t))$ for all $t \in \mathbb{R}_+ \cup \{\infty\}$ and all $i \geq 2$.

We observe that for a given time $t \geq 0$, $\tau_\sigma^1(t)$ is the first switching time greater than t , $\tau_\sigma^2(t)$ is the second switching time greater than t , etc. We also define, for convenience, $\tau_\sigma^0(t) = t$ for all $t \geq 0$.

Definition 4.1 *Given a bounded trajectory $(x, \sigma) \in \mathcal{T}_a$, a point $(\xi, \gamma) \in \mathbb{R}^n \times \Gamma$ belongs to $\Omega^\sharp(x, \sigma)$ if there exists a strictly increasing and unbounded sequence $\{s_k\} \subset \mathbb{R}_+$ such that*

1. $\lim_{k \rightarrow \infty} \tau_\sigma^1(s_k) - s_k = r$, $0 < r \leq \infty$;
2. $\lim_{k \rightarrow \infty} x(s_k) = \xi$ and $\lim_{k \rightarrow \infty} \sigma(s_k) = \gamma$.

We observe that in the case when N_σ is finite, $\Omega^\sharp(x, \sigma) = \Omega(x) \times \{\gamma_{N_\sigma}\}$.

The following lemma shows the relation between $\Omega(x)$ and $\Omega^\sharp(x, \sigma)$.

Lemma 4.1 *Let (x, σ) be a bounded trajectory belonging to \mathcal{T}_a . Then $\Omega(x) = \pi_1(\Omega^\sharp(x, \sigma))$.*

Proof. We only prove the case when σ has infinitely many switching times since the other case is trivial.

Suppose that $\sigma \in \mathcal{S}_a[\tau_D, N_0]$ for some $\tau_D > 0$ and $N_0 \in \mathbb{N}$ and that σ has infinitely many switching times. We first note that the inclusion $\pi_1(\Omega^\sharp(x, \sigma)) \subseteq \Omega(x)$ readily follows from the definition of $\Omega^\sharp(x, \sigma)$.

In order to prove that $\Omega(x) \subseteq \pi_1(\Omega^\sharp(x, \sigma))$ let $\xi \in \Omega(x)$. Then there exists a strictly increasing and unbounded sequence of times $\{s_k\}$ such that $x(s_k) \rightarrow \xi$.

Let $i \geq 0$ be the first integer such that

$$\limsup_{k \rightarrow \infty} \tau_\sigma^{i+1}(s_k) - s_k = r, \text{ with } 0 < r \leq \infty.$$

Such an integer exists and verifies $i \leq N_0$ since, due to the definition of $\mathcal{S}_a[\tau_D, N_0]$, $\tau_\sigma^{N_0+1}(s_k) - s_k \geq \tau_D$ for all $k \geq 1$. Let $\{s_{k_j}\}$ be a subsequence of $\{s_k\}$ so that $\lim_{j \rightarrow \infty} \tau_\sigma^{i+1}(s_{k_j}) - s_{k_j} = r$. Then, i) $\lim_{j \rightarrow \infty} \tau_\sigma^l(s_{k_j}) - s_{k_j} = 0$ for all $0 \leq l \leq i$.

Consider the sequence $\{\sigma(\tau_\sigma^i(s_{k_j}))\}_{j=1}^\infty$; as Γ is compact, there is a subsequence $\{\sigma(\tau_\sigma^i(s_{k_{j_l}}))\}_{l=1}^\infty$ which converges to some $\gamma \in \Gamma$.

We claim that $(\xi, \gamma) \in \Omega^\sharp(x, \sigma)$. In order to prove the claim, consider the unbounded sequence $\{s'_l = \tau_\sigma^i(s_{k_{j_l}})\}$. From i) and the facts that $\lim_{j \rightarrow \infty} \tau_\sigma^{i+1}(s_{k_j}) - s_{k_j} = r$ and $\tau_\sigma^1(s'_l) = \tau_\sigma^1(\tau_\sigma^i(s_{k_{j_l}})) = \tau_\sigma^{i+1}(s_{k_{j_l}})$, we have that $\lim_{l \rightarrow \infty} \tau_\sigma^1(s'_l) - s'_l = r > 0$.

We note that by construction

$$\lim_{l \rightarrow \infty} \sigma(s'_l) = \lim_{l \rightarrow \infty} \sigma(\tau_\sigma^i(s_{k_{j_l}})) = \gamma.$$

Finally, taking into account that:

- $\lim_{l \rightarrow \infty} s'_l - s_{k_{j_l}} = 0$;
- x is uniformly continuous on \mathbb{R}_+ since \dot{x} is essentially bounded (x is bounded, f is continuous and Γ is compact);
- $\lim_{k \rightarrow \infty} x(s_k) = \xi$;

it follows that $\lim_{l \rightarrow \infty} x(s'_l) = \xi$. Thus $(\xi, \gamma) \in \Omega^\sharp(x, \sigma)$ and thereby $\xi \in \pi_1(\Omega^\sharp(x, \sigma))$. Then $\Omega(x) \subseteq \pi_1(\Omega^\sharp(x, \sigma))$ and the lemma follows. \blacksquare

The next result shows that, under suitable hypotheses, the set $\Omega^\sharp(x, \sigma)$ corresponding to a trajectory in $\mathcal{T}_V \cap \mathcal{T}[\chi] \cap \mathcal{T}_a[\tau_D, N_0]$ is weakly-invariant w.r.t. $\mathcal{T}_V^* \cap \mathcal{T}[\chi] \cap \mathcal{T}_a[\tau_D, N_0]$.

Proposition 4.1 Let χ be a closed covering of \mathbb{R}^n and let $V : \Omega \times \Gamma \rightarrow \mathbb{R}$ be a continuous function. Suppose that (x, σ) is a trajectory belonging to $\mathcal{T}_V \cap \mathcal{T}[\chi] \cap \mathcal{T}_a[\tau_D, N_0]$, $\tau_D > 0, N_0 \in \mathbb{N}$, such that for some compact set $B \subset \Omega$, $x(t) \in B$ for all $t \geq 0$. Then $\Omega^\sharp(x, \sigma)$ is weakly-invariant w.r.t. $\mathcal{T}_V^* \cap \mathcal{T}[\chi] \cap \mathcal{T}_a[\tau_D, N_0]$.

Proof. Let $(\xi, \gamma) \in \Omega^\sharp(x, \sigma)$. Then there exists a strictly increasing and unbounded sequence $\{s_k\}$ which verifies 1. and 2. of Definition 4.1. Let $\sigma_k(\cdot) = \sigma(\cdot + s_k)$ and $x_k(\cdot) = x(\cdot + s_k)$. As χ, V and the sequence $\{(x_k, \sigma_k)\}$ are as in the hypotheses of Lemma A.2 in the Appendix, there exist a subsequence (x_{k_l}, σ_{k_l}) and a trajectory $(x^*, \sigma^*) \in \mathcal{T}_V \cap \mathcal{T}[\chi] \cap \mathcal{T}_a[\tau_D, N_0]$ such that $\{x_{k_l}\}$ converges uniformly to x^* on compact subsets of \mathbb{R}_+ and $\{\sigma_{k_l}\}$ converges to σ^* a.e. on \mathbb{R}^+ . Due to Lemma A.1 in the Appendix, we can also assume without loss of generality that $\{\sigma_{k_l}\}$ also verifies condition 2 of that lemma with σ^* in place of σ .

The proof is completed provided we show that $(x^*(0), \sigma^*(0)) = (\xi, \gamma)$, $(x^*(t), \sigma^*(t)) \in \Omega^\sharp(x, \sigma)$ for all $t \geq 0$ and $(x^*, \sigma^*) \in \mathcal{T}_V^*$.

Let us prove first that $(x^*(0), \sigma^*(0)) = (\xi, \gamma)$. From the fact that $x_k(0) = x(s_k)$, 2. of Definition 4.1 and the convergence of $\{x_{k_l}\}$ to x^* , we have that $x^*(0) = \xi$.

According to 2. of Lemma A.1, there exists a sequence $\{r_l\} \subset \mathbb{R}_+$ such that $\lim_{l \rightarrow \infty} r_l = 0$, $\lim_{l \rightarrow \infty} \tau_{\sigma_{k_l}}^1(r_l) - r_l > 0$ and $\lim_{l \rightarrow \infty} \sigma_{k_l}(r_l) = \sigma^*(0)$. From item 1. of Definition 4.1 and the fact that $\tau_{\sigma_k}^1(0) = \tau_\sigma^1(s_k) - s_k$ for all $k \in \mathbb{N}$, it follows that $\lim_{l \rightarrow \infty} \tau_{\sigma_{k_l}}^1(0) > 0$. Then $r_l < \tau_{\sigma_{k_l}}^1(0)$ for l large enough and, therefore, $\sigma^*(0) = \lim_{l \rightarrow \infty} \sigma_{k_l}(r_l) = \lim_{l \rightarrow \infty} \sigma_{k_l}(0) = \lim_{l \rightarrow \infty} \sigma(s_{k_l}) = \gamma$.

Next we prove that $(x^*(t), \sigma^*(t)) \in \Omega^\sharp(x, \sigma)$ for all $t > 0$.

Let $t > 0$ and let $\{r_l\}$ be a sequence as in 2. of Lemma A.1. Consider the unbounded sequence $\{s'_l\}$, defined by $s'_l = r_l + s_{k_l}$, which we can suppose, without loss of generality, strictly increasing. Due to the fact that $\tau_\sigma^1(s'_l) = \tau_{\sigma_{k_l}}^1(r_l) + s_{k_l}$, we have that $\lim_{l \rightarrow \infty} \tau_\sigma^1(s'_l) - s'_l > 0$. So $\{s'_l\}$ satisfies condition 1. of Definition 4.1.

From 2. of Lemma A.1, we have that $\lim_{l \rightarrow \infty} \sigma(s'_l) = \lim_{l \rightarrow \infty} \sigma_{k_l}(r_l) = \sigma^*(t)$. On the other hand, from the uniform convergence of $\{x_{k_l}\}$ to x^* on compact sets and the continuity of x^* we have that $\lim_{l \rightarrow \infty} x(s'_l) = \lim_{l \rightarrow \infty} x_{k_l}(r_l) = x^*(t)$. Hence $(x(s'_l), \sigma(s'_l)) \rightarrow (x^*(t), \sigma^*(t))$ as $l \rightarrow \infty$ and thereby $(x^*(t), \sigma^*(t)) \in \Omega^\sharp(x, \sigma)$.

Finally, we prove that $(x^*, \sigma^*) \in \mathcal{T}_V^*$. Since $(x^*, \sigma^*) \in \mathcal{T}_V$ it suffices to prove that for any pair of times t, t' with $t < t'$ and $\sigma^*(t) = \sigma^*(t')$, $V(x^*(t), \sigma^*(t)) \leq V(x^*(t'), \sigma^*(t'))$.

Let t, t' be a pair of times such that $t < t'$ and $\sigma^*(t) = \sigma^*(t') = \gamma$. Since $\lim_{l \rightarrow \infty} \sigma_{k_l}(t) = \sigma^*(t)$ a.e. on \mathbb{R}_+ , and σ^* is piecewise constant and right continuous, there exists a pair of non-increasing sequences $\{\tau_i\}, \{\tau'_i\}$ so that

- $\tau_i < \tau'_i$ for all i ;
- $\tau_i \searrow t$ and $\tau'_i \searrow t'$;
- for every i , $\lim_{l \rightarrow \infty} \sigma_{k_l}(\tau_i) = \sigma^*(\tau_i) = \gamma$ and $\lim_{l \rightarrow \infty} \sigma_{k_l}(\tau'_i) = \sigma^*(\tau'_i) = \gamma$.

Fix i . Since Γ is finite, there exists l^* such that $\sigma_{k_l}(\tau_i) = \sigma_{k_l}(\tau'_i) = \gamma$ for all $l \geq l^*$. Fix $l' \geq l^*$. As $\{s_{k_l}\}$ is unbounded, we have that $\tau_i + s_{k_l} > \tau'_i + s_{k_{l'}}$ for l large enough, say $l \geq l_0$. Then, for $l \geq \max\{l_0, l^*\}$, $\tau_i + s_{k_l} > \tau'_i + s_{k_{l'}}$ and $\sigma(\tau_i + s_{k_l}) = \sigma_{k_l}(\tau_i) = \sigma_{k_{l'}}(\tau'_i) = \sigma(\tau'_i + s_{k_{l'}})$. In consequence,

$$\begin{aligned} V(x_{k_{l'}}(\tau'_i), \sigma_{k_{l'}}(\tau'_i)) &= V(x(\tau'_i + s_{k_{l'}}), \sigma(\tau'_i + s_{k_{l'}})) \geq \\ &= V(x(\tau_i + s_{k_l}), \sigma(\tau_i + s_{k_l})) = V(x_{k_l}(\tau_i), \sigma_{k_l}(\tau_i)). \end{aligned}$$

From the latter, after taking limit as $l \rightarrow \infty$, we get $V(x_{k_{l'}}(\tau'_i), \sigma_{k_{l'}}(\tau'_i)) \geq V(x^*(\tau_i), \sigma^*(\tau_i))$ and from this, letting $l' \rightarrow \infty$, we obtain $V(x^*(\tau'_i), \sigma^*(\tau'_i)) \geq V(x^*(\tau_i), \sigma^*(\tau_i))$. Finally, from the continuity of V and x^* and the right continuity of σ^* , letting $i \rightarrow \infty$ it follows that $V(x^*(t), \sigma^*(t)) \leq V(x^*(t'), \sigma^*(t'))$. ■

Now we are ready to prove the extension of LaSalle's invariance principle given in Theorem 2.4.

Proof of Theorem 2.4. It readily follows from Proposition 4.1 and Lemma 4.1. In fact, from Proposition 4.1 we have that $\Omega^\sharp(x, \sigma) \subseteq M$. Thus, from Lemma 4.1, we deduce that $\Omega(x) \subseteq \pi_1(M)$ and therefore that $x(t)$ tends to $\pi_1(M)$ as $t \rightarrow \infty$. ■

5 Proof of Theorem 2.2

Proof of Theorem 2.2. Since from Theorem 2.1 we know that \mathcal{T}' is uniform stable, we only have to prove the remaining statements.

Let $(x, \sigma) \in \mathcal{T}'$. Since \mathcal{T}' is uniform stable, it follows that (x, σ) is bounded and therefore evolves into some compact subset B of $\Omega = \mathbb{R}^n$. By applying Theorem 2.4 we deduce that $x(t)$ tends to $\pi_1(M) = \{0\}$ and the global asymptotic stability of \mathcal{T}' follows.

Suppose now that $\mathcal{T}' \subseteq \mathcal{T}_a[\tau_D, N_0]$. Since V is a weak Lyapunov-like function for the family of trajectories $\mathcal{T}^* = \mathcal{T}_V \cap \mathcal{T}[\chi] \cap \mathcal{T}_a[\tau_D, N_0]$, and $\mathcal{T}' \subseteq \mathcal{T}^*$, it suffices to prove that \mathcal{T}^* is globally uniformly asymptotically stable.

Since we have already proved that \mathcal{T}^* is globally asymptotically stable, and in particular uniformly stable, the global uniform asymptotic stability of \mathcal{T}^* will be established if we show that \mathcal{T}^* verifies (a) of Remark 2.1.

As \mathcal{T}^* is invariant by time translations, i.e., for all $s \geq 0$, $(x(\cdot + s), \sigma(\cdot + s)) \in \mathcal{T}^*$ if $(x, \sigma) \in \mathcal{T}^*$, in order to prove (a) of Remark 2.1 it is sufficient to show that \mathcal{T}^* verifies the weaker condition:

(*) for each $R > 0$ and each $\varepsilon > 0$, there exists $T \geq 0$ such that for all $(x, \sigma) \in \mathcal{T}^*$,

$$|x(0)| < R \implies |x(t)| < \varepsilon \quad \forall t \geq T.$$

Suppose that (*) does not hold. Then there exist $\varepsilon_0 > 0$, $\eta_0 > 0$, a sequence of trajectories $\{(x_k, \sigma_k)\} \subset \mathcal{T}^*$ and an increasing and unbounded sequence of times $\{\tau_k\}$ such that $|x_k(0)| \leq \eta_0$ and $|x_k(\tau_k)| \geq \varepsilon_0$ for all k .

Since $\{(x_k, \sigma_k)\}$ is uniformly bounded, from Lemma A.2 we know that there exists a subsequence $\{(x_{k_l}, \sigma_{k_l})\}$ and a trajectory $(x^*, \sigma^*) \in \mathcal{T}^*$ such that $\{x_{k_l}\}$ converges to x^* uniformly on compact sets. Let $\varepsilon' = \alpha^{-1}(\varepsilon_0/2)$, with α as in 1. of Definition 2.1. As x^* converges to 0 as $t \rightarrow \infty$, there exists a time $T > 0$ such that $|x^*(T)| < \varepsilon'$. Since $\{x_{k_l}\}$ converges to x^* uniformly on compact sets, $|x_{k_l}(T)| < \varepsilon'$ for l large enough. Then, due to the uniform stability of \mathcal{T}^* , we have that, for l large enough and $t \geq T$,

$$|x_{k_l}(t)| \leq \alpha(|x_{k_l}(T)|) \leq \alpha(\varepsilon') = \frac{\varepsilon_0}{2},$$

which is a contradiction. ■

6 Conclusions

In this paper we have presented an extension of LaSalle's invariance principle for switched nonlinear systems assuming that the family of subsystems is finite and that the switching signals have a positive average dwell-time. This extension enabled us to obtain some asymptotic stability criteria for this class of systems. Examples were presented that show the application of our results to cases either intractable with some of the previously mentioned results or upon which those results give no conclusive answers. In addition, results about the compactness of the trajectories of the systems involved were exhibited that not only were instrumental used in the proof of some results along the paper, but are important by their own.

Finally, we point out that extensions of LaSalle's principle for switched nonlinear systems in the case that Γ is infinite and also integral invariance principles for the same class of switched systems have been already obtained and are currently under preparation for publication.

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Appendix

A Some compactness results of trajectories of switched systems

In this Appendix we will show that, under suitable hypotheses, certain families of trajectories of system (1) enjoy a certain kind of sequential compactness.

We say that a sequence $\{(x_k, \sigma_k)\}$ of trajectories of (1) is uniformly bounded if there exists $M \geq 0$ such that for all k , $|x_k(t)| \leq M$ for all $t \geq 0$.

Definition A.1 A family \mathcal{T}^* of maximal trajectories of (1) has the **SC** property if for every uniformly bounded sequence $\{(x_k, \sigma_k)\} \subset \mathcal{T}^*$ there exist a subsequence $\{(x_{k_l}, \sigma_{k_l})\}$ and a trajectory $(x^*, \sigma^*) \in \mathcal{T}^*$ such that $\{x_{k_l}\}$ converges to x^* uniformly on compact sets of \mathbb{R}_+ and $\lim_{l \rightarrow \infty} \sigma_{k_l}(t) = \sigma^*(t)$ a.e. on \mathbb{R}_+ .

Proposition A.1 Assume that χ is a closed covering of \mathbb{R}^n . Then $\mathcal{T}[\chi] \cap \mathcal{T}_a[\tau_D, N_0]$ has the **SC** property for all $\tau_D > 0$ and all $N_0 \in \mathbb{N}$.

The following lemma is used in the proof of Proposition A.1 and in some parts of section 4.

Lemma A.1 Let $\{\sigma_k\}$ be a sequence of switching signals in $\mathcal{S}_a[\tau_D, N_0]$ with $\tau_D > 0$ and $N_0 \in \mathbb{N}$.

Then there exist a subsequence $\{\sigma_{k_l}\}$ and a switching signal $\sigma \in \mathcal{S}_a[\tau_D, N_0]$ such that

1. $\lim_{l \rightarrow \infty} \sigma_{k_l}(t) = \sigma(t)$ for almost all $t \geq 0$;
2. for each $t \in \mathbb{R}_+$ there exists a sequence of positive times $\{r_l\}_{l=1}^{\infty}$ such that $\lim_{l \rightarrow \infty} r_l = t$,

$$\lim_{l \rightarrow \infty} \sigma_{k_l}(r_l) = \sigma(t) \quad \text{and} \quad \lim_{l \rightarrow \infty} \tau_{\sigma_{k_l}}^1(r_l) - r_l > 0.$$

Proof. Let $\overline{\mathbb{R}_+} = \mathbb{R}_+ \cup \{\infty\}$ be the one-point compactification of \mathbb{R}_+ , which we recall is a compact metric space, and let $\mathcal{K} = (\overline{\mathbb{R}_+} \times \Gamma)^{\mathbb{N}_0}$ be the set of all the sequences $p = \{(t_i, \gamma_i) : t_i \in \overline{\mathbb{R}_+}, \gamma_i \in \Gamma, i \in \mathbb{N}_0\}$ endowed with the product topology. We note that since \mathcal{K} is the Cartesian product of a countable number of compact metric spaces, it is metrizable (see [4], Theorem 7.2 on p. 190) and compact (see [4], Theorem 1.4 on p. 224).

For each $k \in \mathbb{N}$, let $\{t_i^k\}_{i=0}^\infty$ be the sequence of switching times associated to the switching signal σ_k and let $\{\gamma_i^k\}_{i=0}^\infty$ be the sequence of points of Γ defined by $\gamma_i^k = \sigma_k(t_i^k)$. Observe that when N_{σ_k} is finite we have, according to the convention above, that $\gamma_i^k = \gamma^*$ and $t_i^k = \infty$ for every $i > N_{\sigma_k}$. Let $p_k = \{p_k(i) = (t_i^k, \gamma_i^k)\}_{i=0}^\infty \in \mathcal{K}$.

As \mathcal{K} is a compact metric space, there exists a subsequence $\{p_{k_l}\}_{l=1}^\infty$ which converges, say to $p = \{p(i) = (t_i, \gamma_i)\}_{i=0}^\infty$, i.e., for each $i \in \mathbb{N}_0$, $t_i^{k_l} \rightarrow t_i$ and $\gamma_i^{k_l} \rightarrow \gamma_i$ as $l \rightarrow \infty$.

Since for each $k \in \mathbb{N}$, $\{t_i^k\}_{i=0}^\infty$ is nondecreasing and $t_0^k = 0$, it readily follows that $\{t_i\}_{i=0}^\infty$ is nondecreasing and $t_0 = 0$.

We claim that:

- (a) For every open interval (a, b) , with $a < b$, the number of indexes $i \in \mathbb{N}$ such that $t_i \in (a, b)$, is bounded by $N_0 + (b - a)/\tau_D$;
- (b) the number of indexes i such that $t_i = 0$ is at most $N_0 + 1$.

We will only prove (a) since (b) can be proved in a similar way.

Proof of (a). Suppose on the contrary that there are $r > N_0 + (b - a)/\tau_D$ indexes, say i_1, \dots, i_r , such that $t_{i_j} \in (a, b)$ for $j = 1, \dots, r$. Since $t_{i_j}^{k_l} \rightarrow t_{i_j}$ for each $j = 1, \dots, r$, we have that $t_{i_j}^{k_l} \in (a, b)$ for all $j = 1, \dots, r$ if l is large enough, which contradicts the fact that σ_k belongs to $\mathcal{S}_a[\tau_D, N_0]$.

In order to define σ as in the thesis of the lemma, let $\{i_j\}_{j=0}^N$, with $N \leq \infty$, be the unique subsequence of \mathbb{N}_0 that verifies:

- $0 = t_0 = \dots = t_{i_0}$ and $t_{i_0+1} > 0$;
- $t_{i_j+1} = \dots = t_{i_{j+1}}$ and $t_{i_j+1} < t_{i_{j+1}+1}$ for all $0 \leq j < N$;
- $t_{i_N+1} = \infty$ when $N < \infty$.

Note that this subsequence is well defined due to (a) and (b).

Now, let $\sigma : \mathbb{R}_+ \rightarrow \Gamma$ be the switching signal defined by: $\sigma(t) = \gamma_{i_j}$ for all $t \in [t_{i_j}, t_{i_{j+1}})$ and all $j \leq N$. From (a) it readily follows that $\sigma \in \mathcal{S}_a[\tau_D, N_0]$.

Now we proceed to prove 1. and 2. of the thesis of the lemma. We will consider two cases.

Case I. $t \notin \{t_{i_j}\}_{j=0}^N$.

Let $j^* \in \mathbb{N}_0$ so that $t_{i_{j^*}} < t < t_{i_{j^*+1}}$. As $\lim_{l \rightarrow \infty} t_{i_{j^*}}^{k_l} = t_{i_{j^*}}$ and $\lim_{l \rightarrow \infty} t_{i_{j^*+1}}^{k_l} = t_{i_{j^*+1}}$, $t_{i_{j^*}}^{k_l} < t < t_{i_{j^*+1}}^{k_l}$ for l large enough, say $l \geq L$. Therefore, for $l \geq L$, $\sigma_{k_l}(t) = \sigma_{k_l}(t_{i_{j^*}}^{k_l}) = \gamma_{i_{j^*}}^{k_l}$ and, consequently,

$$\lim_{l \rightarrow \infty} \sigma_{k_l}(t) = \lim_{l \rightarrow \infty} \gamma_{i_{j^*}}^{k_l} = \gamma_{i_{j^*}} = \sigma(t). \quad (3)$$

As $\{t_{i_j}\}_{j=0}^N$ is a set of measure zero, the latter shows 1).

From the fact that $t_{i_{j^*}}^{k_l} < t < t_{i_{j^*+1}}^{k_l}$ for $l \geq L$, we also have that

$$\lim_{l \rightarrow \infty} \tau_{\sigma_{k_l}}^1(t) - t = \lim_{l \rightarrow \infty} t_{i_{j^*+1}}^{k_l} - t = t_{i_{j^*+1}} - t > 0,$$

which shows that 2) holds with the sequence $\{r_l\}$ defined by $r_l = t$ for all $l \in \mathbb{N}$.

Case II. $t = t_{i_{j^*}}$ for some $j^* \in \mathbb{N}_0$.

By using arguments similar to those used in the preceding case, we have that

$$\lim_{l \rightarrow \infty} \tau_{\sigma_{k_l}}^1(t) - t_{i_{j^*}}^{k_l} = \lim_{l \rightarrow \infty} t_{i_{j^*+1}}^{k_l} - t_{i_{j^*}}^{k_l} = t_{i_{j^*+1}} - t_{i_{j^*}} > 0,$$

and that $\lim_{l \rightarrow \infty} \sigma(t_{i_{j^*}}^{k_l}) = \sigma(t)$.

Consequently, item 2. holds with the sequence $\{r_l\}$ defined by $r_l = t_{i_{j^*}}^{k_l}$ for all $l \in \mathbb{N}$. ■

Remark A.1 It is worth mentioning that it is not necessary that Γ be a finite set for the thesis of Lemma A.1 to hold. In fact, as can be easily seen from its proof, it suffices that Γ be a compact metric space.

Remark A.2 A result on compactness of switching signals, proved with arguments different to ours, has recently appeared in [7]. That result (Theorem 1 of that paper) states that given $\Gamma = \{1, \dots, N\}$, the set of switching signals with a *fixed* dwell-time $\tau_D > 0$ is a compact subset of the metric space (\mathcal{S}, d) where the metric d is defined by

$$d(u, v) = \sum_{n=1}^{\infty} 2^{-n} \int_0^n |u(s) - v(s)| ds,$$

for all u and v in \mathcal{S} .

By using Lemma A.1 one can generalize Theorem 1 in [7] to average dwell-time signals (the *fixed* dwell-time hypothesis is essential in the proof given in [7]). In fact, if we consider the metric space $(\mathcal{S}_a[\tau_D, N_0], d)$ with d the metric above, the compactness of $\mathcal{S}_a[\tau_D, N_0]$ follows from Lemma A.1 and the application of Lebesgue's Dominated Convergence Theorem. Moreover, since as pointed out in Remark A.1, Lemma A.1 holds for a compact metric space (Γ, ρ) , $\mathcal{S}_a[\tau_D, N_0]$ is compact with the metric

$$d(u, v) = \sum_{n=1}^{\infty} 2^{-n} \int_0^n \rho(u(s), v(s)) ds,$$

for all u and v in \mathcal{S} .

Proof of Proposition A.1. As $\{(x_k, \sigma_k)\}$ is bounded there exists $M \geq 0$ such that $|x_k(t)| \leq M$ for all $t \geq 0$ and all $k \in \mathbb{N}$. Let $M' = \max_{|\xi| \leq M, \gamma \in \Gamma} |f(\xi, \gamma)|$. Then, for every positive integer k , $|\dot{x}_k(t)| \leq M'$ for almost all $t \in [0, +\infty)$. In consequence $\{x_k\}$ is equibounded and equicontinuous. Then, applying the Arzela-Ascoli Theorem we deduce the existence of a subsequence $\{x_{k_l}\}$ and a continuous function $x^* : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ such that $\{x_{k_l}\}$ converges to x^* uniformly on compact subsets of \mathbb{R}_+ .

Consider the subsequence $\{\sigma_{k_l}\}$. Due to Lemma A.1 we can suppose without loss of generality that there exists $\sigma^* \in \mathcal{S}_a[\tau_D, N_0]$ such that $\lim_{l \rightarrow +\infty} \sigma_{k_l}(t) = \sigma^*(t)$ for almost all $t \geq 0$.

We claim that x^* is a solution of (1) corresponding to σ^* and, in consequence, $(x^*, \sigma^*) \in \mathcal{T}_a[\tau_D, N_0]$.

Let $t > 0$. Then

$$\begin{aligned} x^*(t) &= \lim_{l \rightarrow +\infty} x_{k_l}(t) = \lim_{l \rightarrow +\infty} \left(x_{k_l}(0) + \int_0^t f(x_{k_l}(s), \sigma_{k_l}(s)) ds \right) \\ &= x^*(0) + \lim_{l \rightarrow +\infty} \int_0^t f(x_{k_l}(s), \sigma_{k_l}(s)) ds. \end{aligned}$$

As $|f(x_{k_l}(s), \sigma_{k_l}(s))| \leq M'$ for all $s \in [0, t]$ and $\lim_{l \rightarrow +\infty} f(x_{k_l}(s), \sigma_{k_l}(s)) = f(x^*(s), \sigma^*(s))$ for almost all $s \in [0, t]$, applying the Lebesgue Dominated Convergence Theorem we have that

$$\lim_{l \rightarrow +\infty} \int_0^t f(x_{k_l}(s), \sigma_{k_l}(s)) ds = \int_0^t f(x^*(s), \sigma^*(s)) ds,$$

and, consequently,

$$x^*(t) = x^*(0) + \int_0^t f(x^*(s), \sigma^*(s)) ds.$$

It only remains to show that (x^*, σ^*) belongs to $\mathcal{T}[\chi]$. Let $t \geq 0$ so that $\lim_{l \rightarrow \infty} \sigma_{k_l}(t) = \sigma^*(t) = j$. As Γ is finite, there exists $l^* > 0$ such that $\sigma_{k_l}(t) = j$ for all $l \geq l^*$. Note that due to the definition of $\mathcal{T}[\chi]$, we also have that $x_{k_l}(t) \in \chi_j$ for all $l \geq l^*$. As $\{x_{k_l}(t)\}$ converges to $x^*(t)$ and χ_j is closed, we deduce that $x^*(t) \in \chi_j$. In consequence,

$$x^*(t) \in \chi_{\sigma^*(t)} \quad \text{for almost all } t \geq 0. \quad (4)$$

Now let $t \geq 0$ be arbitrary. Due to (4) there exists a sequence $\{s_k\}$ which converges to t so that $t \leq s_k$, $\sigma^*(s_k) = \sigma^*(t)$ and $x^*(s_k) \in \chi_{\sigma^*(t)}$ for all k . Then, from the continuity of x^* and the fact that $\chi_{\sigma^*(t)}$ is closed, we have that $\lim_{k \rightarrow \infty} x^*(s_k) = x^*(t) \in \chi_{\sigma^*(t)}$ and the proof is completed. \blacksquare

Lemma A.2 Assume that χ is a closed covering of \mathbb{R}^n and $V : \Omega \times \Gamma \rightarrow \mathbb{R}$, with Ω an open subset of \mathbb{R}^n , is continuous. Let $\{(x_k, \sigma_k)\}$ be a sequence of maximal trajectories of (1) belonging to $\mathcal{T}_V \cap \mathcal{T}[\chi] \cap \mathcal{T}_a[\tau_D, N_0]$, $\tau_D > 0$, $N_0 \in \mathbb{N}$, and suppose that there exists a compact subset $B \subset \Omega$ such that $x_k(t) \in B$ for all $t \geq 0$ and all k .

Then there exist a subsequence $\{(x_{k_l}, \sigma_{k_l})\}$ and a maximal trajectory $(x^*, \sigma^*) \in \mathcal{T}_V \cap \mathcal{T}[\chi] \cap \mathcal{T}_a[\tau_D, N_0]$ such that $\{x_{k_l}\}$ converges to x^* uniformly on compact sets of \mathbb{R}_+ and $\lim_{l \rightarrow \infty} \sigma_{k_l}(t) = \sigma^*(t)$ a.e. on \mathbb{R}_+ . \square

Proof. Since $\{(x_k, \sigma_k)\}$ is uniformly bounded, from Proposition A.1 there exist a subsequence $\{(x_{k_l}, \sigma_{k_l})\}$ and a maximal trajectory $(x^*, \sigma^*) \in \mathcal{T}[\chi] \cap \mathcal{T}_a[\tau_D, N_0]$ such that $\{x_{k_l}\}$ converges to x^* uniformly on compact sets of \mathbb{R}_+ and $\lim_{l \rightarrow \infty} \sigma_{k_l}(t) = \sigma^*(t)$ a.e. on \mathbb{R}_+ . Therefore the lemma follows provided $(x^*, \sigma^*) \in \mathcal{T}_V$. As $x^*(t) \in B \subset \Omega$ for all $t \in \mathbb{R}_+$, it only remains to prove that for all pair of times t, t' , with $t < t'$ and $\sigma^*(t) = \sigma^*(t')$, $V(x^*(t), \sigma^*(t)) \geq V(x^*(t'), \sigma^*(t'))$.

Let t, t' be a pair of times such that $t < t'$ and $\sigma^*(t) = \sigma^*(t') = \gamma$. Since $\lim_{l \rightarrow \infty} \sigma_{k_l}(t) = \sigma^*(t)$ a.e. on \mathbb{R}_+ , and σ^* is piecewise constant and continuous from the right, there exists a pair of non-increasing sequences $\{s_i\}, \{s'_i\}$ so that

- $s_i < s'_i$ for all i ;
- $s_i \searrow t$ and $s'_i \searrow t'$;
- for every i , $\lim_{l \rightarrow \infty} \sigma_{k_l}(s_i) = \sigma^*(s_i) = \gamma$ and $\lim_{l \rightarrow \infty} \sigma_{k_l}(s'_i) = \sigma^*(s'_i) = \gamma$.

Fix i . Since Γ is finite, we have that $\sigma_{k_l}^*(s_i) = \sigma_{k_l}^*(s'_i) = \gamma$ for l large enough, say $l \geq l^*$. Then, from the definition of \mathcal{T}_V and the fact that $s_i < s'_i$, it follows that $V(x_{k_l}(s_i), \sigma_{k_l}(s_i)) \geq V(x_{k_l}(s'_i), \sigma_{k_l}(s'_i))$ for all $l \geq l^*$. In consequence, by taking limit as $l \rightarrow \infty$ we obtain that $V(x^*(s_i), \sigma^*(s_i)) \geq V(x^*(s'_i), \sigma^*(s'_i))$ and, *a posteriori*, letting $i \rightarrow \infty$ we have that $V(x^*(t), \sigma^*(t)) \geq V(x^*(t'), \sigma^*(t'))$. ■