Invariance Results for Constrained Switched Systems

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***Abstract*— In this paper we address invariance principles for nonlinear switched systems with otherwise arbitrary compact index set and with constrained switchings. We present an extension of LaSalle's invariance principle for these systems and derive by using detectability notions some convergence and asymptotic stability criteria. These results enable to take into account in the analysis of stability not only state-dependent constraints but also to treat the case in which the switching logic has memory, i.e., the active subsystem only can switch to a prescribed subset of subsystems.**

***Index Terms*— Switched systems, detectability, stability, convergence, invariance**

I. INTRODUCTION

A switched system is a dynamical system that consists of a family of subsystems and a logical rule (time or state dependent) that orchestrates switching between them. Although switched systems may look simple, their behavior may be very complicated, being a classical example the fact that a trajectory obtained by switching among asymptotically stable subsystems may be divergent (see [9]). Although the stability of switched systems under arbitrary switching laws can be assured by the existence of a common Lyapunov function (CLF) for all the switching modes ([9], [11]), in practical applications many switched systems do not share a CLF. Nevertheless, they may be stable under restricted switching signals. Restrictions on the set of admisible switching signals of a certain switched system arise naturally from physical constraints of the system, from design strategies (e.g. discontinuous control feedback laws), or from the knowledge about possible switching logic of the switched system, e.g., partitions of the state space and their induced switching rules.The framework of multiple Lyapunov functions (MLF) is the usual one in the stability analysis of switched systems with constrained switchings.In this method each switching mode may have its own Lyapunov function (see [4] and references therein), however, some additional conditions are necessary to assure the value of each Lyapunov function on its corresponding mode will decrease. Sufficient conditions for asymptotic stability of switched systems with MLF can be found in [4], [9] and references therein. When the derivative of a candidate Lyapunov function with respect to each mode is only *non-positive*, the asymptotic stability of the switched system can be derived by using one of the various extensions of LaSalle's invariance principle for switched systems: Hespanha in [6] introduced an invariance principle for switched linear systems under *persistently dwell-time* switching signals and in [7] Hespanha *et al.* extended some of those results to a family of nonlinear systems. Bacciotti and Mazzi in [1] presented an invariance principle for switched systems with *dwell-time* signals. An invariance principle for switched nonlinear systems with *average dwell-time* signals that satisfy *state-dependent* constraints was derived by Mancilla-Aguilar and García in [13] from the sequential compactness of particular classes of solutions to switched systems. Based on invariance results for hybrid systems ([14]), Goebel *et al.* ([5]) obtained recently invariance results for switched systems under various types of switching signals. Lee and Jiang in [8] gave a generalized version of Krasovskii-LaSalle Theorem for time-varying switched systems. Under certain ergodicity conditions on the switching signal, some stability results were obtained in [3] and in [15].

Most of the invariance results for switched systems already published only consider restrictions originated by the timing of the switchings or by the state dependence of it. Nevertheless there is also an important restriction to take into account: the fact that not all the subsystems may be accessible from a particular one, i.e. when the switching logic has memory (see [12]). This restriction is clearly exhibited, for example, in switched systems which are the continuous portion of a hybrid automaton (see [4], [10]). In this regard, the invariance principles developed for hybrid systems in [10] and in [14] could be useful in the analysis of switched systems with this class of restriction in the switchings.

In this paper we present an invariance principle for switched systems that takes into account this additional restriction. This invariance principle differs from those obtained in the mentioned papers in that the index sets that we consider may be arbitrary compact sets and the type of invariance considered is both backward and forward. From this invariance principle we derive some new convergence and stability criteria that extend some previously obtained results. These criteria involve some detectability conditions on the functions which bound the derivatives of Lyapunov-like functions. The article unfolds as follows. Section II contains the basic definitions. In Section III we present an invariance principle for switched systems with constrained switching. Convergence and stability results are given in Section IV. Finally Section V contains some conclusions.

II. BASIC DEFINITIONS

In this work we consider switched systems described by

$$\dot{x} = f(x, \sigma) \tag{1}$$

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where x takes values in $\mathbb{R}^n$, $\sigma : \mathbb{R} \to \Gamma$, with Γ a compact metric space, is a *switching signal*, *i.e.*, σ is piecewise constant (it has at most a finite number of jumps in each compact interval) and is continuous from the right and $f : \mathrm{dom}(f) \to \mathbb{R}^n$, where $\mathrm{dom}(f)$ is a closed subset of $\mathbb{R}^n \times \Gamma$, is continuous. For each $\gamma \in \Gamma$, let $\chi_\gamma = \{\xi \in \mathbb{R}^n : (\xi, \gamma) \in \mathrm{dom}(f)\}$ and $f_\gamma : \chi_\gamma \to \mathbb{R}^n$ defined by $f_\gamma(\xi) = f(\xi, \gamma)$; then χ_γ is closed and f_γ is continuous for any $\gamma \in \Gamma$. We note that when Γ is finite, the last conditions imply that $\mathrm{dom}(f)$ is closed and that f is continuous on $\mathrm{dom}(f)$.

We will denote by $\mathcal{S}$ the set of all the switching signals. Given $\sigma \in \mathcal{S}$, a solution of (1) corresponding to σ is a locally absolutely continuous function $x : I_x \to \mathbb{R}^n$, with $I_x \subset \mathbb{R}$ a nonempty interval, such that $(x(t), \sigma(t)) \in \mathrm{dom}(f)$ for all $t \in I_x$ and $\dot{x}(t) = f(x(t), \sigma(t))$ for almost all $t \in I_x$. The solution x is complete if $I_x = \mathbb{R}$ and forward complete if $\mathbb{R}_{\geq 0} \subset I_x$. A pair (x, σ) is a trajectory of (1) if $\sigma \in \mathcal{S}$ and x is a solution of (1) corresponding to σ. The trajectory is complete or forward complete if x is complete or forward complete, respectively. Given a subset $\mathcal{O}$ of $\mathbb{R}^n$, we say that the trajectory (x, σ) is precompact relative to $\mathcal{O}$ if there exists a compact set $B \subset \mathcal{O}$ such that $x(t) \in B$ for all $t \in I_x$. When $\mathcal{O} = \mathbb{R}^n$ we simply say that (x, σ) is precompact.

Remark 2.1: Note that we do not suppose that $\mathrm{dom}(f) = \mathbb{R}^n \times \Gamma$. This fact enables us to take into account in the analysis of the asymptotic behavior of a given trajectory (x, σ) of (1) some kind of state-dependent constraints which the trajectory under study must satisfy. In some situations we are not interested in the analysis of all the forward complete trajectories (x, σ) of a switched system (1) (with $\mathrm{dom}(f) = \mathbb{R}^n \times \Gamma$) but only of those that verify the constraint

$$x(t) \in \chi_{\sigma(t)} \quad \text{for all } t \in I_x, \tag{2}$$

where $\{\chi_\gamma : \gamma \in \Gamma\}$ is a collection of subsets of $\mathbb{R}^n$. If we consider the map $\tilde{f}$, which is the restriction of f to the set $\mathrm{dom}(\tilde{f}) = \{(\xi, \gamma) : \xi \in \chi_\gamma\}$, and $\mathrm{dom}(\tilde{f})$ is closed in $\mathbb{R}^n \times \Gamma$, then the set of trajectories (x, σ) of (1) which verify (2) coincides with the set of trajectories of

$$\dot{x} = \tilde{f}(x, \sigma). \tag{3}$$

It must be pointed out that in this way we can consider the system as if its switching is state-independent, and focus on the restrictions imposed to it by the timing of the discontinuities of the switching signal and/or by the accessibility to certain subsystems from another ones.

In this paper we consider forward complete solutions of (1) corresponding to switching signals σ which belong to particular subclasses of $\mathcal{S}$. Let $\Lambda(\sigma)$ be the set of times where σ has a jump (switching time). Following ([6]) we say that $\sigma \in \mathcal{S}$ has a dwell-time $\tau_D > 0$ if $|t - t'| \geq \tau_D$ for any pair $t, t' \in \Lambda(\sigma)$ such that $t \neq t'$.

A switching signal σ has an average dwell-time $\tau_D > 0$ and a chatter bound $N_0 \in \mathbb{N}$ if the number of switching times of σ in any open finite interval $(\tau_1, \tau_2) \subset \mathbb{R}$ is bounded by $N_0 + (\tau_2 - \tau_1)/\tau_D$, i.e. $\mathrm{card}(\Lambda(\sigma) \cap (\tau_1, \tau_2)) \leq N_0 + (\tau_2 - \tau_1)/\tau_D$.

We denote by $\mathcal{S}_a[\tau_D, N_0]$ the set of all the switching signals which have an average dwell-time $\tau_D > 0$ and a chatter bound $N_0 \in \mathbb{N}$. $\mathcal{T}_a[\tau_D, N_0]$ denotes the set of all the complete trajectories (x, σ) of (1) with $\sigma \in \mathcal{S}_a[\tau_D, N_0]$. Let $\mathcal{S}_a = \bigcup_{\tau_D > 0, N_0 > 0} \mathcal{S}_a[\tau_D, N_0]$ and $\mathcal{T}_a = \bigcup_{\tau_D > 0, N_0 > 0} \mathcal{T}_a[\tau_D, N_0]$. We note that the set of switching signals σ which have a dwell-time $\tau_D > 0$ coincides with $\mathcal{S}_a[\tau_D, 1] := \mathcal{S}_d[\tau_D]$. We denote by $\mathcal{T}_d[\tau_D]$ the set of all the complete trajectories (x, σ) of (1) with $\sigma \in \mathcal{S}_d[\tau_D]$. Let $\mathcal{S}_d = \bigcup_{\tau_D > 0} \mathcal{S}_d[\tau_D]$ and $\mathcal{T}_d = \bigcup_{\tau_D > 0} \mathcal{T}_d[\tau_D]$.

The following family of trajectories is introduced in order to take into account the case in which the switching logic has memory, i.e. when a subsystem corresponding to an index $\gamma \in \Gamma$ only can switch to subsystems corresponding to modes γ' that belong to a certain subset $\Gamma_\gamma \subset \Gamma$. Given a set value-map $H : \Gamma \rightsquigarrow \Gamma$, $\mathcal{S}^H$ is the set of all the switching signals σ which verify the condition $\sigma(t) \in H(\sigma(t^-))$ for every time $t \in \Lambda(\sigma)$. Here $\sigma(t^-) = \lim_{s \to t^-} \sigma(s)$. $\mathcal{T}^H$ denotes the set of all the complete trajectories (x, σ) with $\sigma \in \mathcal{S}^H$. This class of switching signals enable us, for example, to model the restrictions imposed by the discrete process of a hybrid system whose continuous portion is as in (1) (see [4]).

III. Invariance results for trajectories which satisfy a dwell-time condition

In what follows we study the asymptotic behavior of a precompact forward complete trajectory (x, σ) of (1) with $\sigma \in \mathcal{S}_a$.

We recall that a point $\xi \in \mathbb{R}^n$ belongs to $\Omega(x)$, the ω-limit set of $x : I_x \to \mathbb{R}^n$, with $\mathbb{R}_{\geq 0} \subset I_x$, if there exists a strictly increasing sequence of times $\{s_k\}$ with $\lim_{k \to \infty} s_k = \infty$ and $\lim_{k \to \infty} x(s_k) = \xi$. The ω-limit set $\Omega(x)$ is always closed and, when x evolves in a compact set of $\mathbb{R}^n$, it is nonempty, compact and $x \to \Omega(x)$ (for a set $M \subset \mathbb{R}^n$, $x \to M$ if $\lim_{t \to +\infty} d(x(t), M) = 0$, where $d(\xi, M) = \inf_{\nu \in M} |\nu - \xi|$).

As was done in [13], we will associate to each precompact forward complete trajectory (x, σ) of (1) with $\sigma \in \mathcal{S}_a$, the nonempty set $\Omega^\sharp(x, \sigma) \subset \mathbb{R}^n \times \Gamma$ defined by

Definition 1: Given a precompact forward complete trajectory (x, σ) of (1) with $\sigma \in \mathcal{S}_a$, a point $(\xi, \gamma) \in \mathbb{R}^n \times \Gamma$ belongs to $\Omega^\sharp(x, \sigma)$ if there exists a strictly increasing and unbounded sequence $\{s_k\} \subset \mathbb{R}_{\geq 0}$ such that

1) $\lim_{k \to \infty} \tau_\sigma^1(s_k) - s_k = r$, $0 < r \leq \infty$;
2) $\lim_{k \to \infty} x(s_k) = \xi$ and $\lim_{k \to \infty} \sigma(s_k) = \gamma$.

Here, for any $t \in \mathbb{R}$, $\tau_\sigma^1(t) = \inf\{s \in \Lambda(\sigma) : t < s\}$ if $\{s \in \Lambda(\sigma) : t < s\} \neq \emptyset$ and $\tau_\sigma^1(t) = +\infty$ in other case ($\tau_\sigma^1(t)$ is the first switching time greater than t.)

The following relation between $\Omega(x)$ and $\Omega^\sharp(x, \sigma)$ holds.

Lemma 1: Let (x, σ) be a forward complete trajectory of (1) with $\sigma \in \mathcal{S}_a$ that is precompact relative to $\mathcal{O} \subset \mathbb{R}^n$. Then $\Omega^\sharp(x, \sigma) \subset \mathrm{dom}(f) \cap (\mathcal{O} \times \Gamma)$ and $\Omega(x) = \pi_1(\Omega^\sharp(x, \sigma))$, where $\pi_1 : \mathbb{R}^n \times \Gamma \to \mathbb{R}^n$ is the projection onto the first component.

Proof. That $\Omega^\sharp(x, \sigma) \subset \mathrm{dom}(f) \cap (\mathcal{O} \times \Gamma)$ follows from the fact that $(x(t), \sigma(t))$ belongs to a compact subset of $\mathrm{dom}(f) \cap (\mathcal{O} \times \Gamma)$ for all $t \in I_x$ and from the definition of $\Omega^\sharp(x, \sigma)$. The proof of the other assertion follows *mutatis* *mutandis* from the proof of Lemma 4.1 in [13]. ∎

The set $\Omega^\sharp(x,\sigma)$ enjoys certain kind of invariance property. First we introduce the following

Definition 2: Given a family $\mathcal{T}^*$ of complete trajectories of (1), we say that a nonempty subset $M \subset \mathbb{R}^n \times \Gamma$ is weakly-invariant with respect to $\mathcal{T}^*$ if for each $(\xi,\gamma) \in M$ there is a trajectory $(x,\sigma) \in \mathcal{T}^*$ such that $x(0) = \xi$, $\sigma(0) = \gamma$ and $(x(t),\sigma(t)) \in M$ for all $t \in \mathbb{R}$.

This notion of weak invariance differs from the one introduced in [13], in that the last one involves only forward invariance while the introduced here also involves backward invariance.

The next proposition, whose proof we omit, establishes the weak invariance of $\Omega^\sharp(x,\sigma)$ with respect to certain classes of complete trajectories under study in this paper.

Proposition 3.1: Let (x,σ) be a precompact forward complete trajectory of (1).

1) If $\sigma \in \mathcal{S}_a[\tau_D, N_0]$ for some $\tau_D > 0$ and $N_0 \in \mathbb{N}$, then $\Omega^\sharp(x,\sigma)$ is weakly invariant with respect to $\mathcal{T}_a[\tau_D, N_0]$.
2) If $\sigma \in \mathcal{S}_d[\tau_D] \cap \mathcal{S}^H$ for some $\tau_D > 0$ and some $H : \Gamma \rightsquigarrow \Gamma$ such that $\mathrm{Graph}(H) = \{(\gamma,\gamma') \in \Gamma\times\Gamma :\ \gamma' \in H(\gamma)\}$ is closed, then $\Omega^\sharp(x,\sigma)$ is weakly invariant with respect to $\mathcal{T}_d[\tau_D] \cap \mathcal{T}^H$. ∎

Remark 3.1: At first glance, it would seem more natural to consider for a given precompact forward complete trajectory (x,σ) of (1) its ω-limit set $\Omega(x,\sigma)$ instead of $\Omega^\sharp(x,\sigma) \subset \Omega(x,\sigma)$. Nevertheless, there exist forward complete trajectories (x,σ) of (1) with $\sigma \in \mathcal{S}_a$ such that $\Omega(x,\sigma)$ is not weakly-invariant for any family of trajectories of that switched system.

Next, we present an invariance result which involves the existence of a function V which is nonincreasing along a trajectory of (1). For a set $\mathcal{O} \subset \mathbb{R}^n$ and $\gamma \in \Gamma$ let $\mathcal{O}_\gamma = \mathcal{O} \cap \chi_\gamma$.

Assumption 1: The forward complete trajectory (x,σ) of (1) verifies the following:

1) there exists a function $V : \mathcal{O}\times\Gamma \to \mathbb{R}$, with $\mathcal{O}$ an open subset of $\mathbb{R}^n$ such that V is continuous on $\mathrm{dom}(f) \cap (\mathcal{O} \times \Gamma)$ and for all $\gamma \in \Gamma$, $V(\cdot,\gamma)$ is differentiable at every $\xi \in \mathcal{O}_\gamma$;
2) $v(t) = V(x(t),\sigma(t))$ is nonincreasing on $[0,\infty)$.

In what follows, we denote $Z = \{(\xi,\gamma) \in \mathrm{dom}(f)\cap(\mathcal{O}\times\Gamma) : \frac{\partial V}{\partial \xi}(\xi,\gamma) f_\gamma(\xi) = 0\}$.

Theorem 1: Suppose that (x,σ) is a forward complete trajectory of (1) for which Assumption 1 holds. Suppose in addition that (x,σ) is precompact relative to $\mathcal{O}$.

1) If σ belongs to $\mathcal{S}_a[\tau_D, N_0]$ for some $\tau_D > 0$ and some $N_0 \in \mathbb{N}$, then there exists $c \in \mathbb{R}$ such that $x \to \pi_1(M(c))$, where $M(c)$ is the maximal weakly invariant set w.r.t. $\mathcal{T}_a[\tau_D, N_0]$ contained in $\mathrm{dom}(f) \cap V^{-1}(c) \cap Z$.
2) If σ belongs to $\mathcal{S}_d[\tau_D] \cap \mathcal{S}^H$ for some $\tau_D > 0$ and some $H : \Gamma \rightsquigarrow \Gamma$, with $\mathrm{Graph}(H)$ closed, then there exists $c \in \mathbb{R}$ such that $x \to \pi_1(M^*(c))$, where $M^*(c)$ is the maximal weakly invariant set w.r.t. $\mathcal{T}_d[\tau_D] \cap \mathcal{T}^H$ contained in $\mathrm{dom}(f) \cap V^{-1}(c) \cap Z$.

Proof. Since $\Omega^\sharp(x,\sigma)$ is weakly-invariant w.r.t. $\mathcal{T}_a[\tau_D, N_0]$ or $\mathcal{T}_d[\tau_D] \cap \mathcal{T}^H$ when, respectively, $\sigma \in \mathcal{S}_a[\tau_D, N_0]$ or $\sigma \in \mathcal{S}_d[\tau_D] \cap \mathcal{S}^H$, $\Omega^\sharp(x,\sigma) \subset \mathrm{dom}(f) \cap (\mathcal{O} \times \Gamma)$ and $x \to \pi_1(\Omega^\sharp(x,\sigma))$, we only have to prove that $\Omega^\sharp(x,\sigma) \subset V^{-1}(c) \cap Z$ for some $c \in \mathbb{R}$. As (x,σ) is precompact relative to $\mathcal{O}$, there exists a compact set $B \subset \mathcal{O}$ such that $x(t) \in B$ for all $t \in I_x$. Therefore $(x(t),\sigma(t))$ belongs to the compact set $\mathrm{dom}(f) \cap (B \times \Gamma)$ for all $t \in I_x$. Thus $v(t)$ is bounded, since V is continuous on $\mathrm{dom}(f) \cap (\mathcal{O} \times \Gamma)$, and nonincreasing by hypothesis; in consequence there exists $\lim_{t\to+\infty} v(t) = c$.

Let $(\xi,\gamma) \in \Omega^\sharp(x,\sigma)$. Then there exists a strictly increasing and unbounded sequence $\{s_k\}$ which verifies 1. and 2. of Definition 1. We have that $(x(s_k),\sigma(s_k)) \to (\xi,\gamma)$ as $k \to \infty$. In consequence $c = \lim_{k\to\infty} v(s_k) = \lim_{k\to\infty} V(x(s_k),\sigma(s_k)) = V(\xi,\gamma)$ and $(\xi,\gamma) \in V^{-1}(c)$. Now we show that (ξ,γ) also belongs to Z. Since $\Omega^\sharp(x,\sigma) \subset V^{-1}(c)$ is weakly invariant w.r.t. $\mathcal{T}_a[\tau_D, N_0]$ (resp. $\mathcal{T}_d[\tau_D] \cap \mathcal{T}^H$) there exists $(x^*,\sigma^*) \in \mathcal{T}_a[\tau_D, N_0]$ (resp. $\mathcal{T}_d[\tau_D] \cap \mathcal{T}^H$) such that $(x^*(0),\sigma^*(0)) = (\xi,\gamma)$ and $(x^*(t),\sigma^*(t)) \in \Omega^\sharp(x,\sigma)$ for all $t \in \mathbb{R}$. So, $V(x^*(t),\sigma^*(t)) = c$ for all $t \in \mathbb{R}$. In particular, since $\sigma^*(t) = \gamma$ on $[0,\tau)$ for τ small enough, we have that $V(x^*(t),\gamma) = c$ on $[0,\tau)$. Therefore $\frac{\partial V}{\partial \xi}(x^*(0),\gamma) f_\gamma(x^*(0)) = 0$, and $(\xi,\gamma) \in Z$. ∎

When Γ is a finite set, we identify Γ with the finite subset of $\mathbb{N}$, $\{1,\ldots,N\}$, where $N = \mathrm{card}(\Gamma)$. In this case we can relax the nonincreasing condition in Assumption 1 as follows.

Assumption 2: The forward complete trajectory (x,σ) of (1) verifies the following:

1) There exists a function $V : \mathcal{O} \times \Gamma \to \mathbb{R}$, with $\mathcal{O}$ an open set of $\mathbb{R}^n$, such that for all $\gamma \in \Gamma$, $V(\cdot,\gamma)$ is differentiable at every $\xi \in \mathcal{O}_\gamma$;
2) for each $\gamma \in \Gamma$, $v(t) = V(x(t),\sigma(t))$ is nonincreasing on the set $T_\gamma = \sigma^{-1}(\gamma) \cap [0,+\infty)$.

Assumptions of this kind are standard when the stability analysis of the zero solution of a switched system is performed with the help of multiple Lyapunov functions (see [9], [4]).

We note that when Γ is finite, any function V which verifies 1. of Assumption 2 also verifies condition 1. of Assumption 1.

Theorem 2: Suppose that Γ is finite and that (x,σ) is a forward complete trajectory of (1) for which Assumption 2 holds. Suppose in addition that (x,σ) is precompact relative to $\mathcal{O}$.

1) If σ belongs to $\mathcal{S}_a[\tau_D, N_0]$ for some $\tau_D > 0$ and some $N_0 \in \mathbb{N}$, then there exists $\vec{c} \in \mathbb{R}^N$ such that $x \to \pi_1(M(\vec{c}))$, where $M(\vec{c})$ is the maximal weakly invariant set w.r.t. $\mathcal{T}_a[\tau_D, N_0]$ contained in $\cup_{\gamma\in\Gamma}\{(\xi,\gamma) \in \mathrm{dom}(f)\cap(\mathcal{O}\times\Gamma) :\ V(\xi,\gamma) = c_\gamma\} \cap Z$.
2) If σ belongs to $\mathcal{S}_d[\tau_D] \cap \mathcal{S}^H$ for some $\tau_D > 0$ and some $H : \Gamma \rightsquigarrow \Gamma$, then there exists $\vec{c} \in \mathbb{R}^N$ such that $x \to \pi_1(M^*(\vec{c}))$, where $M^*(\vec{c})$ is the maximal

weakly invariant set w.r.t. $\mathcal{T}_d[\tau_D] \cap \mathcal{T}^H$ contained in $\cup_{\gamma\in\Gamma}\{(\xi,\gamma) \in \mathrm{dom}(f) \cap (\mathcal{O}\times\Gamma) :\ V(\xi,\gamma) = c_\gamma\} \cap Z$.

Proof. For $\gamma \in \Gamma$ we define c_γ as follows:

- $c_\gamma = 0$ if $\sigma^{-1}(\gamma) \cap [0,\infty)$ is bounded;
- $c_\gamma = \lim_{t\to+\infty,\, t\in T_\gamma} v(t)$ if $\sigma^{-1}(\gamma) \cap [0,\infty)$ is unbounded.

We note that the limit exists since v is non-increasing and bounded on T_γ.

Reasoning as in the proof of Theorem 1, and taking into account that $\mathrm{Graph}(H)$ is closed since Γ is finite, it suffices to show that $\Omega^\sharp(x,\sigma) \subset \{(\xi,\gamma) \in \mathrm{dom}(f) \cap (\mathcal{O}\times\Gamma) : V(\xi,\gamma) = c_\gamma\} \cap Z$, in order to prove the thesis of the theorem.

Let $(\xi,\gamma) \in \Omega^\sharp(x,\sigma) \subset \mathrm{dom}(f)$. Then there exists a strictly increasing and unbounded sequence $\{s_k\}$ which verifies 1. and 2. of Definition 1. Since $\sigma(s_k) \to \gamma$ and Γ is a finite set, $\sigma(s_k) = \gamma$ for k large enough and for those k, $s_k \in T_\gamma$. It follows that $V(x(s_k),\gamma) \to c_\gamma$ as $k \to \infty$ and in consequence, $V(\xi,\gamma) = c_\gamma$ and $(\xi,\gamma) \in \cup_{\gamma'\in\Gamma}\{(\xi',\gamma') \in \mathrm{dom}(f) \cap (\mathcal{O}\times\Gamma) :\ V(\xi',\gamma') = c_{\gamma'}\}$. The fact that $(\xi,\gamma) \in Z$ can be proved in the same way as in the proof of Theorem 1. ∎

Remark 3.2: If in Assumptions 1 and 2 we remove the hypothesis on the differentiability of V, theorems 1 and 2 remain valid removing the set Z from the statements of theses theorems.

Remark 3.3: Some invariance results for switched systems reported in the literature can be derived from Theorem 2. In particular, [1, Thms. 1 and 2], [13, Prop. 4.1] and [5, Corol. 5.6].

IV. CONVERGENCE AND STABILITY RESULTS

In this section we derive, from the invariance principles presented in the previous section, some convergence and stability results for switching systems with constrained switchings.

A. Convergence results

In the sequel we will consider the following assumptions

Assumption 3: $f_\gamma(0) = 0$ for all $\gamma \in \Gamma$ such that $0 \in \chi_\gamma$.

Assumption 4: The initial value problem $\dot{x} = f_\gamma(x)$, $x(0) = 0$ has a unique solution for every $\gamma \in \Gamma$ such that $0 \in \chi_\gamma$.

Let us introduce some definitions. Given a map $g : \mathbb{R}^n \to \mathbb{R}^n$, a function $W : \mathbb{R}^n \to \mathbb{R}$ and a subset $\mathcal{V} \subset \mathbb{R}^n$ in which g is continuous, we say that for a given τ ($\tau > 0$ or $\tau = \infty$) a point $\xi \in \mathcal{V}$ is *τ-forward-indistinguishable from 0 on $\mathcal{V}$* (resp. *τ-backward-indistinguishable from 0 on $\mathcal{V}$*), if there exists a solution $\varphi : [0,\tau] \to \mathcal{V}$ (resp. $\varphi : [-\tau, 0] \to \mathcal{V}$) of $\dot{x} = g(x)$ such that $\varphi(0) = \xi$ and $W(\varphi(t)) = 0$ for all $t \in [0,\tau]$ (resp. $t \in [-\tau, 0]$).

We denote by $\mathcal{V}^f(g,W,\tau)$ ($\mathcal{V}^b(g,W,\tau)$) the set of points $\xi \in \mathcal{V}$ that are τ-forward(backward)-indistinguishable from 0 on $\mathcal{V}$. We also consider the sets $\mathcal{V}^f(g,W) = \bigcup_{\tau>0}\mathcal{V}^f(g,W,\tau)$, $\mathcal{V}^b(g,W) = \bigcup_{\tau>0}\mathcal{V}^b(g,W,\tau)$ and $\mathcal{V}(g,W) = \mathcal{V}^f(g,W) \cup \mathcal{V}^b(g,W)$.

We say that the pair (g,W) is *zero-state small-time detectable in $\mathcal{V}$* if $\mathcal{V}(g,W) \subset \{0\}$. This detectability condition is equivalent to the following one: for each $a < b$, if $\varphi : [a,b] \to \mathcal{V}$ is a solution of $\dot{x} = g(x)$ such that $W(\varphi(t)) = 0$ for all $t \in [a,b]$ then $\varphi(t) = 0$ for all $t \in [a,b]$. This last condition was considered in ([5]) where it was referred to as observability. We note that $\mathcal{V}(g,W) \subset \{0\} \Leftrightarrow \mathcal{V}^f(g,W) \subset \{0\} \Leftrightarrow \mathcal{V}^b(g,W) \subset \{0\}$.

Remark 4.1:

1) We do not assume that $W(0) = 0$ in the above definitions. So, 0 might not necessarily belong to $\mathcal{V}^b(g,W,\tau)$ or $\mathcal{V}^f(g,W,\tau)$ even in the case when $0 \in \mathcal{V}$. This posibility is convenient for our purposes.
2) The set $\mathcal{V}^f(g,W,\infty)$ ($\mathcal{V}^b(g,W,\infty)$) coincides with the maximal weakly forward(backward) invariant set with respect to g contained in the set $\{\xi \in \mathcal{V} : W(\xi) = 0\}$. We recall that a subset $K \subset \mathbb{R}^n$ is weakly forward(backward) invariant with respect to g if for each $\xi \in K$ there exists a solution $\varphi : [0,\infty) \to \mathbb{R}^n$ ($\varphi : (-\infty, 0] \to \mathbb{R}^n$) of $\dot{x} = g(x)$ such that $\varphi(0) = \xi$ and $\varphi(t) \in K$ for all $t \ge 0$ ($t \le 0$).
3) When g is a linear function and W is a quadratic form, i.e., $g(\xi) = A\xi$ and $W(\xi) = \xi^T C^T C \xi$, with A and C matrices, (g,W) is zero-state small-time detectable in $\mathcal{V}$ if $\mathcal{V} \cap \mathcal{U} \subset \{0\}$ being $\mathcal{U}$ the unobservable subspace of (C,A).
4) When g and W are smooth functions we have that
$$\mathcal{V}(g,W) \subset \{\xi \in \mathcal{V} :\ L_g^k W(\xi) = 0\, \forall k \in \mathbb{N}_0\},$$
with $L_g^k W$ the k-th. Lie derivative of W along g.

Let us introduce the following assuptions in order to obtain some convergence results based on the invariance results previously given and the detectability concepts presented.

Assumption 5: The forward complete trajectory (x,σ) of (1) verifies the following:

1) there exist a function $V : \mathcal{O}\times\Gamma \to \mathbb{R}$ as in 1. of Assumption 1 and a family of functions $\{W_\gamma : \mathbb{R}^n \to \mathbb{R}, \gamma \in \Gamma\}$ such that for all $\gamma \in \Gamma$
$$-\frac{\partial V}{\partial \xi}(\xi,\gamma) f_\gamma(\xi) \ge W_\gamma(\xi) \ge 0 \quad \forall \xi \in \mathcal{O}_\gamma. \qquad (4)$$
2) $v(t) = V(x(t),\sigma(t))$ is nonincreasing on $[0,\infty)$.

Assumption 6: For the forward complete trajectory (x,σ) of (1) there exist a function $V : \mathcal{O}\times\Gamma$ which verifies 1. of Assumption 2 and a family of functions $\{W_\gamma : \mathbb{R}^n \to \mathbb{R}\}$ such that (4) holds, and, in addition, for all $\gamma \in \Gamma$ $v(t) = V(x(t),\sigma(t))$ is nonincreasing on $T_\gamma = \sigma^{-1}(\gamma) \cap [0,\infty)$.

Theorem 3: Suppose that assumptions 3 and 4 hold. Let (x,σ) be a forward complete trajectory of (1) with $\sigma \in \mathcal{S}_a$ that verifies Assumption 5.

If (x,σ) is precompact relative to $\mathcal{O}$ and

1) $\mathcal{O}^f_\gamma(f_\gamma, W_\gamma, \infty) \cap \mathcal{O}^b_\gamma(f_\gamma, W_\gamma, \infty) \subset \{0\}$ for every $\gamma \in \Gamma$.
2) $\mathcal{O}^b_\gamma(f_\gamma, W_\gamma) \cap \mathcal{O}^f_{\gamma'}(f_{\gamma'}, W_{\gamma'}) \subset \{0\}, \quad \forall \gamma \ne \gamma'$.

Then $x(t) \to 0$ as $t \to \infty$.

If Γ is finite, the same holds if we suppose that (x,σ) verifies Assumption 6 instead of Assumption 5.

Proof. Since $\sigma \in \mathcal{S}_a$, there exist $\tau_D > 0$ and $N_0 \in \mathbb{N}$ such that $\sigma \in \mathcal{S}_a[\tau_D, N_0]$. Suppose first that (x, σ) verifies (i) Assumption 5 or (ii) Γ is finite and (x, σ) verifies Assumption 6. Let M be the maximal weakly invariant set w.r.t. $\mathcal{T}_a[\tau_D, N_0]$ contained in Z. Since (x, σ) verifies the hypotheses of Theorem 1 when (i) holds and those of Theorem 2 when (ii) holds, and the sets $M(c)$ or $M(\vec{c})$ which appear in these theorems are subsets of M, we have that $x \to \pi_1(M)$. So, it suffices to show that $M \subset \{0\} \times \Gamma$. Let $(\xi, \gamma) \in M$; then there exists a trajectory $(x^*, \sigma^*) \in \mathcal{T}_a[\tau_D, N_0]$ such that $(x^*(0), \sigma^*(0)) = (\xi, \gamma)$ and $(x^*(t), \sigma^*(t)) \in M \subset Z$ for all t. We will consider two cases.

Case 1. σ^* has no switching times, i.e. $\sigma^*(t) = \gamma$ for all $t \in \mathbb{R}$. Thus, $x^*(t) \in \mathcal{O}_\gamma$ and $W_\gamma(\varphi(t)) = 0$ for all $t \in \mathbb{R}$ and, in consequence $x^*(t) = 0$ for all t, since $x^*(t) \in \mathcal{O}^f_\gamma(f_\gamma, W_\gamma, \infty) \cap \mathcal{O}^b_\gamma(f_\gamma, W_\gamma, \infty) \subset \{0\}$.

Case 2. σ^* has a switching time t^*. So, there exists $\tau > 0$ such that, if $\varphi(t) = x^*(t+t^*)$, $\gamma = \sigma^*(t^{*-})$ and $\gamma' = \sigma^*(t^*)$, $\gamma \neq \gamma'$ and

1) φ is solution of $\dot{z} = f_\gamma(z)$ on $[-\tau, 0]$ and φ is solution of $\dot{z} = f_{\gamma'}(z)$ on $[0, \tau]$;
2) $\frac{\partial V}{\partial \xi}(\varphi(t), \gamma) f_\gamma(\varphi(t)) = 0$ on $[-\tau_D, 0]$ and $\frac{\partial V}{\partial \xi}(\varphi(t), \gamma') f_{\gamma'}(\varphi(t)) = 0$ on $[0, \tau]$. Thus, $W_\gamma(\varphi(t)) = 0$ on $[-\tau_D, 0]$ and $W_{\gamma'}(\varphi(t)) = 0$ on $[0, \tau]$.

Therefore $\varphi(0) \in \mathcal{O}^b_\gamma(f_\gamma, W_\gamma, \tau) \cap \mathcal{O}^f_{\gamma'}(f_{\gamma'}, W_{\gamma'}, \tau)$.
Thus, by 2., $x^*(t^*) = \varphi(0) = 0$. That $x^*(0) = 0$ follows from the fact that, due to assumptions 3 and 4 any initial value problem $\dot{z} = f_{\hat{\gamma}}(z)$, $z(0) = 0$ has the unique solution $z(t) \equiv 0$, if $0 \in \mathcal{O}_{\hat{\gamma}}$. ∎

In the case in which Γ is finite and σ belongs to $\mathcal{S}_d \cap \mathcal{S}^H$, hypothesis 2. of Theorem 3 can be weakened as follows. Given a set-valued map $H : \Gamma \rightsquigarrow \Gamma$, a finite sequence $\{\gamma_i\}_{i=1}^m \subset \Gamma$, $m \geq 3$, is a *simple cycle of* H if $\gamma_1 = \gamma_m$, $\gamma_{i+1} \in H(\gamma_i)$ for all $i = 1, \ldots, m-1$ and if $\gamma_i = \gamma_j$ and $i < j$ then $i = 1$ and $j = m$.

Theorem 4: Suppose that Γ is finite, $H : \Gamma \rightsquigarrow \Gamma$, $\tau_D > 0$ and that (x, σ) is a forward complete trajectory of (1) with σ belonging to $\mathcal{S}_d[\tau_D] \cap \mathcal{S}^H$, such that Assumption 6 holds. Suppose in addition that assumptions 3 and 4 hold.

If (x, σ) is precompact relative to $\mathcal{O}$ and

1) $\mathcal{O}^f_\gamma(f_\gamma, W_\gamma, \infty) \subset \{0\}$ for every $\gamma \in \Gamma$ or $\mathcal{O}^b_\gamma(f_\gamma, W_\gamma, \infty) \subset \{0\}$ for every $\gamma \in \Gamma$,
2) for each simple cycle $\{\gamma_i\}_{i=1}^m$ of H there exists $j \in \{1, \ldots, m-1\}$ such that

$$\mathcal{O}^b_{\gamma_j}(f_{\gamma_j}, W_{\gamma_j}, \tau_D) \cap \mathcal{O}^f_{\gamma_{j+1}}(f_{\gamma_{j+1}}, W_{\gamma_{j+1}}, \tau_D) \subset \{0\}, \quad (5)$$

then $x(t) \to 0$ as $t \to \infty$.

Proof. Let M be the maximal weakly invariant set w.r.t. $\mathcal{T}_d[\tau_D] \cap \mathcal{T}^H$ contained in Z. Since (x, σ) verifies the hypotheses of Theorem 2, we have that $x \to \pi_1(M)$. So, it suffices to show that $M \subset \{0\} \times \Gamma$. Let $(\xi, \gamma) \in M$; then there exists a trajectory $(x^*, \sigma^*) \in \mathcal{T}_d[\tau_D] \cap \mathcal{T}^H$ such that $(x^*(0), \sigma^*(0)) = (\xi, \gamma)$ and $(x^*(t), \sigma^*(t)) \in M \subset Z$ for all t. We will consider two cases:

Case 1. σ^* has a finite number of switching times, $t_0 < t_1 < \cdots < t_l$. Suppose first that $\mathcal{O}^f_\gamma(f_\gamma, W_\gamma, \infty) \subset \{0\}$ for every $\gamma \in \Gamma$ and let $\varphi(t) = x^*(t + t_l)$ for $t \geq 0$ and $\gamma_l = \sigma^*(t_l)$. Then φ is a solution of $\dot{z} = f_{\gamma_l}(z)$, $\varphi(t) \in \mathcal{O}_{\gamma_l}$ for all $t \geq 0$ and $\frac{\partial V}{\partial \xi}(\varphi(t), \gamma_l) = 0$ on $[0, \infty)$. Thus, $W_{\gamma_l}(\varphi(t)) = 0$ for all $t \geq 0$ and, in consequence $x^*(t_l) = 0$ since $x^*(t_l) = \varphi(0) \in \mathcal{O}^f_{\gamma_l}(f_\gamma, W_\gamma, \infty) \subset \{0\}$. That $\xi = x^*(0) = 0$ follows from the fact that the unique solution of the initial value problem $\dot{z} = f_\gamma(z)$, $z(0) = 0$ is $z(t) \equiv 0$ for every $\gamma \in \Gamma$ such that $0 \in \mathcal{O}_\gamma$.

In the case when $\mathcal{O}^b_\gamma(f_\gamma, W_\gamma, \infty) \subset \{0\}$ for every $\gamma \in \Gamma$, we proceed in a similar way, but considering $\varphi(t) = x^*(t + t_0)$ for $t \leq 0$.

Case 2. σ^* has an infinite number of switching times, Since $\sigma^* \in \mathcal{S}^H$, there exists a finite sequence of consecutive switching times $\{t_k\}_{k=1}^m$, such that the sequence $\{\gamma_k\}_{k=1}^m$, with $\gamma_k = \sigma^*(t_k)$, is a simple cycle of H. By hypothesis there exists an index $j \in \{1, \ldots, m-1\}$ for which (5) holds. For such j we consider the function $\varphi : [-\tau_D, \tau_D] \to \mathbb{R}^n$ defined by $\varphi(t) = x^*(t + t_{j+1})$. Since $(x^*, \sigma^*) \in \mathcal{T}_d[\tau_D]$ we have that

1) φ is solution of $\dot{z} = f_{\gamma_j}(z)$ on $[-\tau_D, 0]$ and φ is solution of $\dot{z} = f_{\gamma_{j+1}}(z)$ on $[0, \tau_D]$;
2) $\frac{\partial V}{\partial \xi}(\varphi(t), \gamma_j) = 0$ on $[-\tau_D, 0]$ and $\frac{\partial V}{\partial \xi}(\varphi(t), \gamma_{j+1}) = 0$ on $[0, \tau_D]$. Thus, $W_{\gamma_j}(\varphi(t)) = 0$ on $[-\tau_D, 0]$ and $W_{\gamma_{j+1}}(\varphi(t)) = 0$ on $[0, \tau_D]$.

Therefore

$$\varphi(0) \in \mathcal{O}^b_{\gamma_j}(f_{\gamma_j}, W_{\gamma_j}, \tau_D) \cap \mathcal{O}^f_{\gamma_{j+1}}(f_{\gamma_{j+1}}, W_{\gamma_{j+1}}, \tau_D).$$

Thus, by (5), $x^*(t_{j+1}) = \varphi(0) = 0$. By using arguments similar to those of the proof of case 1, we conclude that $\xi = x^*(0) = 0$. ∎

Remark 4.2:

1) From the proofs of theorems 3 and 4 it follows that theses theorems remain valid if, instead of assumptions 3 and 4, we assume that the function V in assumptions 5 and 6 verifies the following: for each $\gamma \in \Gamma$ such that $0 \in \chi_\gamma$, $V(\xi, \gamma) = 0$ and $\xi \in \chi_\gamma \Leftrightarrow \xi = 0$. This condition is fulfilled when, for example, $V(\cdot, \gamma)$ is positive definite on χ_γ for all γ such that $0 \in \chi_\gamma$.
2) If for all $\gamma \in \Gamma$ the pair (f_γ, W_γ) is zero-state small-time detectable in $\mathcal{O}_\gamma$, conditions 1 and 2 of Theorem 3 are fulfilled. Nevertheless, in this case, a careful analysis of the proof of that theorem shows that its conclusions hold without assumptions 3 and 4. So, in the case when every pair(f_γ, W_γ) is zero-state small-time detectable we recover the first conclusion of the convergence result given in [5, Corol.4.10].

B. Stability criteria

Combining the convergence results already presented with well known sufficient Lyapunov conditions for the local stability of a family $\mathcal{T}$ of forward complete trajectories of (1) we can derive some asymptotic stability criteria (see [13] for the definitions of the stability properties of a family of trajectories).

Theorem 5: Let $\mathcal{T}$ be a family of forward complete trajectories of (1) such that for every $(x,\sigma) \in \mathcal{T}$, $\sigma \in \mathcal{S}_a$. Suppose that every trajectory of $\mathcal{T}$ verifies, with the same function V and the same family $\{W_\gamma : \gamma \in \Gamma\}$, Assumption 5 or, in the case when Γ is finite, Asumption 6. Suppose, in addition, that 1. and 2. of Theorem 3 hold and that there exist functions α_1 and α_2 of class $\mathcal{K}$ such that

$$\alpha_1(|\xi|) \leq V(\xi,\gamma) \leq \alpha_2(|\xi|) \quad \forall \xi \in \mathcal{O}_\gamma, \forall \gamma \in \Gamma. \tag{6}$$

Then $\mathcal{T}$ is locally asymptotically stable.

Theorem 6: Suppose that Γ is finite. Let $\mathcal{T}$ be a family of forward complete trajectories of (1) such that for every $(x,\sigma) \in \mathcal{T}$, $\sigma \in \mathcal{S}_d \cap \mathcal{S}^H$, with $H : \Gamma \rightsquigarrow \Gamma$. Suppose that every trajectory of $\mathcal{T}$ verifies, with the same function V and the same family $\{W_\gamma : \gamma \in \Gamma\}$, Assumption 6. Suppose, in addition, that 1. and 2. of Theorem 4 hold and that there exist functions α_1 and α_2 of class $\mathcal{K}$ such that (6) holds.

Then $\mathcal{T}$ is locally asymptotically stable.

Proof of theorems 5 and 6. Since, due to well known results, the hypotheses of both theorems implies that $\mathcal{T}$ is locally uniformly stable (LUS) (see, for example [2] or [9]) we only have to prove that there exists $\eta > 0$ such that for every $(x,\sigma) \in \mathcal{T}$, $|x(0)| < \eta$ implies that $x \to 0$.

Since $\mathcal{T}$ is LUS, there exist $\eta > 0$ and $\rho > 0$ such that, for every $(x,\sigma) \in \mathcal{T}$ such that $|x(0)| < \eta$, $x(t) \in B = \{\xi \in \mathbb{R}^n : |\xi| \leq \rho\} \subset \mathcal{O}$ for all $t \geq 0$. Therefore $(x,\sigma) \in \mathcal{T}$ is precompact relative to $\mathcal{O}$ if $|x(0)| < \eta$. Then, due to 1. of Remark 4.2, to Theorem 3 in the case of Theorem 5 and to Theorem 4 in the case of Theorem 6, we have that $x \to 0$ for any $(x,\sigma) \in \mathcal{T}$ such that $|x(0)| < \eta$, and the local asymptotic stability of $\mathcal{T}$ follows. ∎

1) Example: Consider the switched system with $f : \mathrm{dom}(f) \to \mathbb{R}^2$ where $\mathrm{dom}(f) \subset \mathbb{R}^2 \times \{1,2,3\}$, $\chi_1 = \{\xi : \xi_1 \leq 0 \wedge \xi_1 + \xi_2 \geq 0\} \cup \{\xi : \xi_1 \geq 0 \wedge (\xi_1 + \xi_2 \leq 0 \ \vee -3\xi_1 + \xi_2 \geq 0)\}$, $\chi_2 = \{\xi : \xi_1 \geq 0\} \cup \chi_1$, $\chi_3 = \{\xi : \xi_1 \leq 0\}$ and

$$\begin{aligned} f_1(\xi) &= \begin{bmatrix} \xi_1 + \xi_2 \\ -\xi_2 \end{bmatrix}, f_2(\xi) = \begin{bmatrix} \xi_2 \\ -\xi_1 \end{bmatrix}, \\ f_3(\xi) &= \begin{bmatrix} \xi_2 \\ -2\xi_1 \end{bmatrix}. \end{aligned}$$

Let $V : \mathbb{R}^2 \times \{1,2,3\} \to \mathbb{R}$ given by $V(\xi,1) = V(\xi,2) = |\xi|^2/2$, $V(\xi,3) = \xi_1^2 + \xi_2^2/2$.

Let $W_1(\xi) = \xi_2^2/2$ and $W_2(\xi) = W_3(\xi) = 0$. Then $-\frac{\partial V}{\partial \xi}(\xi,i) f_i(\xi) \geq W_i(\xi)$ on χ_i, $i = 1,2,3$. It is easy to see that the conditions of Assumption 6 hold, that for any $\tau_D > 0$, condition 1) of Theorem 4 also holds and that the $\mathcal{K}$-class functions $\alpha_1(\xi) = |\xi|^2/2$ and $\alpha_2(\xi) = |\xi|^2$ verify the condition (6).

Let H be given by the assignment $H(1) = \{2,3\}$, $H(2) = \{1,3\}$ and $H(3) = \{1\}$. Since $\chi_1^b(f_1, W_1, \tau_D) = \chi_1^f(f_1, W_1, \tau_D) = \{0\}$, it can also be verified that whichever be a simple cycle of H, condition 2) of Theorem 4 holds since for such a cycle there always exists a switching to or from subsystem 1. In consequence the hypotheses of Theorem 6 hold and any class of forward complete trajectories $\mathcal{T} \subset \mathcal{T}_d \cap \mathcal{T}^H$ of this system is (globally) asymptotically stable.

On the other hand, if H is now given by $H(1) = \{2,3\}, H(2) = \{1,3\}, H(3) = \{1,2\}$ condition 2. of Theorem 4 does not hold and we only can assert that $\mathcal{T}$ is uniformly stable.

In fact, consider the simple cycle $2 \to 3 \to 2$, with the switchings taking place on the line $\{\xi : \xi_1 = 0\}$. It is not hard to see that in this case $\chi_2^b(f_2, W_2, \tau_D) \cap \chi_3^f(f_3, W_3, \tau_D) = \{(0,\xi_2) : \xi_2 \leq 0\}$ and that $\chi_3^b(f_3, W_3, \tau_D) \cap \chi_2^f(f_2, W_2, \tau_D) = \{(0,\xi_2) : \xi_2 \geq 0\}$.

V. Conclusions

In this paper we have obtained invariance results for switched systems whose switchings are subjected not only to state-dependent constraints, but also to restrictions on the accessibility of each subsystem from other ones. We also derived from these results convergence and stability criteria for this class of systems. These criteria involve some detectability conditions on the functions which bound the derivatives of the Lyapunov-like functions.

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