SYNTHESIS OF LEB (LINEAR-ELLIPSOIDAL-BOUNDED) REGULATORS AND FILTERS FOR DISCRETE-TIME SYSTEMS

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ABSTRACT

This work is applicable to LTI systems, described by state-variables, whose measurement errors, controls and outputs are to be bounded by upper and lower limits, when the system is driven by bounded plant and measurement noises and bounded disturbances. The methodology has been named LEB for its application to Linear systems, whose variables are contained within Ellipsoidal sets when the disturbances and noises are Bounded.

I: LEB REGULATOR, THE SINGLE-DISTURBANCE CASE

Let's consider a LTI discrete-time system described by the following equations:

$$x(k+1) = A\ x(k) + B\ u(k) + G\ w(k) \tag{1}$$

$$y(k+1) = C\ x(k+1) + v(k+1) \tag{2}$$

where dim(x)=nx1, dim(u)=mx1, dim(w)=ndx1, dim(y)=lx1, dim(v)=lx1 and the matrices are of appropriate dimensions. This system will be controlled by a constant gain regulator that generates a control law $u(k) = K\ x(k)$.

The sets that effectively "contain" the bounded disturbances, the controls and the outputs may be represented by

$$\Omega w = \{w : w'\ (Q)^{-1}\ w \le 1\} \tag{3}$$

$$\Omega u = \{u : u'\ (K\ \Gamma\ K')^{-1}\ u = u'\ (T)^{-1}\ u \le 1\} \tag{4}$$

$$\Omega y = \{y : y'\ (C\ \Gamma\ C')^{-1}\ y = u'\ (S)^{-1}\ u \le 1\} \tag{5}$$

where $umax\ i = |umin\ i| = (Ti)^{1/2}$ for $i \le 1 \le m$ and $ymax\ i = |ymin\ i| = (Si)^{1/2}$ for $1 \le i \le l$. The states may be represented by

$$\Omega x = \{x : x'\ (\Gamma)^{-1}\ x \le 1\} \tag{6}$$

$$\Gamma = (A + B\ K)\ \Gamma\ (A + B\ K)' / (1-\beta) + G\ Q\ G' / \beta \tag{7}$$

and β $(0 < \beta < 1)$ is a parameter that enters in the construction of the bounding ellipsoid.

To find the regulator gain the following set of equations have to be simultaneously satisfied [5]:

$$\Lambda - A' [(\Lambda)^{-1} - B\ (\Lambda K)^{-1}\ B'/(1-\beta)]^{-1}\ A/(1-\beta) + C'\ \Lambda C\ C = 0 \tag{8}$$

$$\theta - (A + B\ K)'\ \theta\ (A + B\ K) / (1-\beta) - G\ G' / \beta = 0 \tag{9}$$

$$K = -[(1 - \beta)\ \Lambda K + B'\ \Lambda\ B]^{-1}\ B'\ \Lambda\ A \tag{10}$$

$$\beta = m / (1 + m) \tag{11}$$

$$m = \left(\frac{\mathrm{trace}(\Lambda\ G\ G')}{\mathrm{trace}[\Lambda\ (A + B\ K)\ \theta\ (A + B\ K)']} \right)^{1/2} \tag{12}$$

$$1 - \mathrm{trace}(\Lambda C\ S) - \mathrm{trace}(\Lambda K\ T) = 0 \tag{13}$$

$$\mathrm{diag}[\ \Lambda K\ (K\ \theta\ K' - T / q)\ ] = 0 \quad (\Lambda K \ge 0) \tag{14}$$

$$\mathrm{diag}[\ \Lambda C\ (C\ \theta\ C' - S / q)\ ] = 0 \quad (\Lambda C \ge 0) \tag{15}$$

where q is the scalar version of Q (ndxnd), nd is the number of disturbances, $\theta = \Gamma/q$ and the matrix Lagrange multipliers are Λ, ΛK and ΛC. The iterative procedure to solve these equations is similar to the procedure used in [3] for the continuous-time case.

II: MODIFIED REGULATORS, EXPANSION OF CONTROL LIMITS

With no filter in the feedback path, the control law is of the type $u(k) = K\ x(k)$. In the presence of a filter, the new control law will be $u1(k) = K1\ xb(k) = K1\ (x(k)+e(k)) = u2(k) + u3(k)$ where K1 is the new modified-LEB-regulator gain, xb is the estimated state variable, x is the real state variable, e is the estimation error $(e(k)=xb(k)-x(k))$, u2 is the control law when Q is not zero (there is plant noise) and R=0 (there is no measurement noise) [1], u3 is the control law when Q=0 (no plant noise) and R is not zero (there is measurement noise) and u1 is the sum of these two control laws.

Using support functions [1,4] for u2 and u3, applying a Holder inequality to the resulting expression and defining a new variable ε $(0<\varepsilon<1)$, a new control u1b can be defined such that u1b is contained inside an ellipsoid $\Omega u1b$ which contains, in turn, the ellipsoid $\Omega u1$, where

$$\Omega u1b : \{u1b :\ u1b'\ (\ T1\ )^{-1}\ u1b \le 1\} \tag{16}$$

$$T1 = K1\ (\Gamma 1 / \varepsilon + \Sigma / (1-\varepsilon))\ K1' \tag{17}$$

where K1 and $\Gamma 1$ are obtained solving the regulator problem for $u1=K1*xb$. Σ is the nxn matrix used to describe the ellipsoid that contains the estimation errors [1].

The bounds for the modified ith control (i=1,2,...,m) are now given by:

$$K1i(\Gamma 1/\varepsilon + \Sigma/(1-\varepsilon))K1i' = T1(i,i) \le T(i,i) \tag{18}$$

or

$$T2(i,i) / \varepsilon + T3(i,i) / (1-\varepsilon) = T1(i,i) \le T(i,i) \tag{19}$$

Then, the maximum excursion for the ith control is now $\pm(T(i,i)^{1/2} + T3(i,i)^{1/2})$ In the presence of noise, there will always be expansions in the maximum achievable controls.

A similar procedure can be followed to define the expansion of output limits (as before, a new variable γ $(0 \le \gamma \le 1)$ will have to be considered). The maximum ith output excursion will be $\pm(S(i,i)^{1/2} + S3(i,i)^{1/2})$ where S3=R.

Note that this approch keeps ε and γ fixed. When they are incorporated in the iterative solution, the equations (8)-(15) are slightly modified and optimum values of ε and γ result.

III: SCALAR EXAMPLE

Given the system represented by $dx/dt = -x + 2u + w$, $y = x + v$, the LEB solution for this continuous-time system is [2]: maximum disturbance = $(qmax)^{1/2}$ = wmax ≅ 17.0 and regulator gain = K = -1.2. The discretized-time system is $x(k+1) = a\,x(k) + b\,u(k) + g\,w(k)$, $y(k+1) = x(k+1) + v(k+1)$, where $a = \exp(-\Delta)$, $g = 1-a$, $b = 2g$ and Δ is the discretizing time step. The results obtained after following the iterative procedure previously described for different Δs between 0 and 1 and choosing maximum control umax = 6 and maximum output ymax = 5, have been presented in [5] and they are briefly summarized here: a) for every Δ there is at least one binding variable -control (umax) or output (ymax)- that reaches the required 100% limit, b) for $0 \le \Delta \le \Delta*$ both limits umax and ymax are simultaneously reached. We'll call $\Delta*$ such time step ($\Delta*$=0.348 in the given example), c) when $0 \le \Delta \le \Delta*$ the numerical values of the maximum disturbance wmax($\Delta*$) and regulator gain K($\Delta*$) agree with the corresponding values of the continuous-time solution (17.0 and -1.2 respectively), d) for every other Δ (i.e. not in that range) the maximum disturbance wmax(Δ) is smaller than wmax($\Delta*$), the regulator gain K(Δ) is less than K($\Delta*$), the maximum output is ymax and the maximum control is smaller than umax.

These facts indicate that, for a continuous-time system whose LEB regulator exists, discretized with an arbitrary time step Δ, the desired maximum controls and outputs umax and ymax will not be simultaneously obtained unless discretizing with $0 \le \Delta \le \Delta*$. Moreover, this circumstance has to be considered when the discrete-time system is already given.

If a filter is used in the feedback path and, according to [1], we consider that the plant noise can be represented by $Q = wmax^2 = 17^2 = 289$, $R = 1.6$ and we fix $\varepsilon \cong \gamma = 0.5$, using the algorithm shown in Section I with the bounds modified as indicated in Section II, we find that the filter gain is $1 - \eta$ ($\eta \to 0$), the maximum measurement error $e = xb - x$ is $\pm(1.6)^{1/2}$, the regulator gain K1=-1.2 and the new maximum disturbance wmax1 = ±11.225. If we simulate this system with disturbances equally distributed between ±wmax, the control bounds will be $\pm(T^{1/2} + T3^{1/2}) = \pm(6.0+1.518)$ and the output bounds will be $\pm(S^{1/2} + S3^{1/2}) \cong \pm(5.0+1.265)$ (Fig.1), clearly exceeding the desired bounds. To recover the original bounds, the disturbance wmax will have to be reduced to wmax1 (Fig.2).

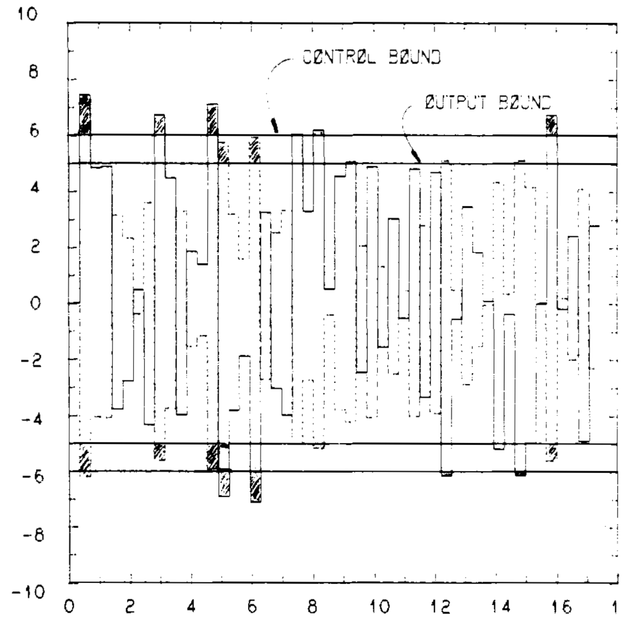

Fig. 1: control u(k) (solid) and output y(k) (dashed) vs. time. Bounds exceeded due to measurement noise.

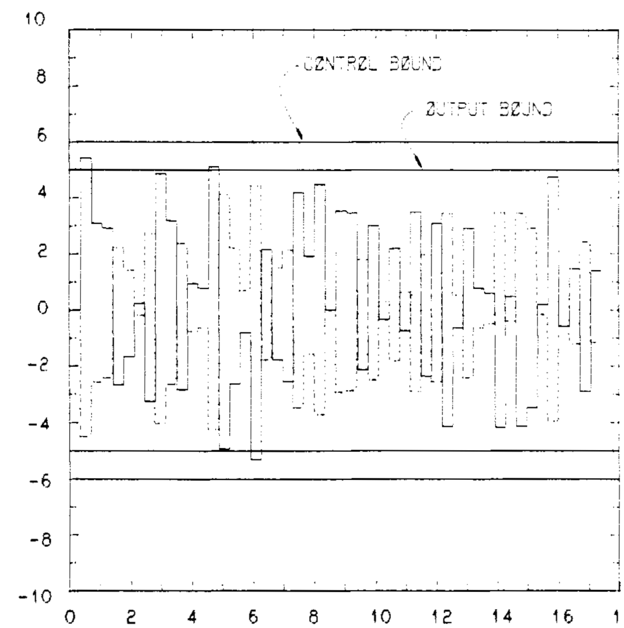

Fig. 2: the original bounds are recovered.

CONCLUSIONS

The inclusion of measurement noise in the synthesis of the LEB regulator and its application to discrete-time systems are extensions to the original algorithms developed for a "set-theoretic" solution to the control and filtering problem. Using an scalar example, some characteristics of this solution were shown. It is particularly important for simulations to note the dependence existing between the maximum reachable controls and outputs and the time step employed.

The treatment of multiple-disturbances, the suboptimality of the regulator found, the robustness of the compensator, the comparison with the LQG synthesis procedure and the shape of the ellipsoid that "contains" the measurement errors as a function of the parameters used to build it have been treated in [4].

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