Uniform Asymptotic Stability of Switched Systems via detectability of reduced control systems*

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***Abstract*— In this paper we present a criterion for the uniform global asymptotic stability of switched nonlinear systems with time/state-dependent switching constraints but with no dwell-time assumptions. This criterion is based on the existence of multiple weak common Lyapunov functions and on a detectability property of a reduced control system.**

I. INTRODUCTION

A switched system is a family of continuous-time dynamical subsystems and a usually time- or state-dependent law —the *switching signal*— that rules the switching between the subsystems. The stability issues of such systems are sources of several interesting phenomena. For example, the switched systems might not inherit the stability properties of its subsystems. In consequence, in the last decades the different stability properties of switched systems have been extensively investigated (see [1]–[3] and references therein).

To determine the uniform asymptotic stability of switched systems is one of the most important issues related to their stability properties (see [1]) and, as in the case of non-switched nonlinear systems, the existence of either strong or weak Lyapunov functions is instrumental in that regard. In fact, the existence of a common strong Lyapunov function implies the global uniform asymptotic stability of switched systems under arbitrary switching (the converse holds for time-invariant switched systems [4]). Consequently, strong Lyapunov functions are highly desirable, although finding them (even for non-switched systems) is not an easy task. On the other hand, the existence of weak Lyapunov functions by itself gives no asymptotic stability guarantee. The fact that only weak common Lyapunov functions (or weak multiple Lyapunov functions) may exist for some switched systems of practical interest, motivated the development of several stability results for switched time-invariant systems: extensions of LaSalle's invariance principle (see [5]–[11]) and other approaches (see [12]–[15]).

The concept of output-persistence of excitation (OPE) plays a fundamental role in proving the uniform asymptotic stability of a switched system, when a weak Lyapunov function is available. Results in this connection for switched time-varying nonlinear systems, are presented in [16]–[21]. An interesting feature of the approach followed in those papers is that it does not involve any dwell-time assumption. The importance of the OPE concept notwithstanding, it is usually difficult to verify this property from its definition, even in the case of time-invariant switched systems. This difficulty motivated the search for sufficient conditions for the OPE property that are easier to check. Those sufficient conditions involve the concept of PE-pairs ([17], [18]) and/or that of weak-zero state detectability (WZSD) ([19], [20]). In [19] it is proved that WZSD implies OPE, while in [20] the authors obtain some criteria for the global uniform asymptotic stability (GUAS) of switched systems assuming the existence of a common zeroing output-system.

On the other hand, when dealing with stability issues of switched systems, it must be taken into account that in addition to the restrictions originated by the timing of the switchings, restrictions on the set of admissible switching signals of a certain switched system arise naturally from physical constraints of the system, from design strategies (e.g. discontinuous control feedback laws), or from the knowledge about possible switching logic of the switched system, e.g., partitions of the state space and their induced switching rules (see [11] and references therein).

In this paper we give sufficient conditions for the WZSD of switched time-invariant systems with a finite number of modes and with time/state-dependent constraints, in terms of the WZSD of an auxiliary control system. In order to do so, we first assume that the switched system has a switched time-invariant output also, and embed both into a control-affine nonlinear system with outputs, whose controls take values in a convex polytope. Next we introduce the concepts of limiting trajectories and reduced control system, and prove that under adequate hypotheses the WZSD of the reduced control system implies that of the switched system. In this way we obtain a criterion for the WZSD of the switched system in terms of the behavior of the solutions of a control system when its outputs are constrained to be identically zero. This criterion incorporates the possible time/state constrains the trajectories of the switching system should satisfy. Finally, we give a criterion for the GUAS of a switched system based on the existence of multiple weak Lyapunov functions and of the WZSD of a reduced control system whose outputs are related with the derivatives of the Lyapunov functions along the trajectories of the subsystems of the switched system. An interesting feature of the criterion is that, as a difference with most criteria based on multiple weak Lyapunov functions, no dwell-time assumptions on the

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switching signals are required. In fact, we derive from it a criterion for GUAS under arbitrary switchings.

Notation: For $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. An indexed family of sets $\chi = \{\chi_i\}_{i=1}^N$ is a closed covering of $\mathbb{R}^n$ if χ_i is a closed subset of $\mathbb{R}^n$ for each i and $\mathbb{R}^n = \cup_{i=1}^N \chi_i$. Given a subset A of $\mathbb{R}^m$, $\mathrm{co}(A)$ is the convex hull of A and $\overline{A}$ denotes its closure. For any matrix B, B' denotes its transpose. $L^1_N(\mathbb{R})$ ($L^\infty_N(\mathbb{R})$) is the set of Lebesgue measurable functions $\varphi : \mathbb{R} \to \mathbb{R}^N$ which are Lebesgue integrable (essentially bounded). We write $\alpha \in \mathcal{K}$ if $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous, strictly increasing and $\alpha(0) = 0$, and $\alpha \in \mathcal{K}_\infty$ if, in addition, α is unbounded. We write $\beta \in \mathcal{KL}$ if $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $\beta(\cdot, t) \in \mathcal{K}_\infty$ for any $t \geq 0$ and, for any fixed $r \geq 0$, $\beta(r, t)$ monotonically decreases to zero as $t \to \infty$.

II. BASIC DEFINITIONS AND PROBLEM STATEMENT

In this work we consider the nonlinear switched system with outputs

$$\dot{x}(t) = f(x(t), \sigma(t)) \quad (1)$$

$$y(t) = h(x(t), \sigma(t)) \quad (2)$$

where for $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$ and $\sigma : \mathbb{R} \to \mathcal{I}$, with $\mathcal{I} = \{1, \ldots, N\}$ the index set, is a *switching signal*, *i.e.* σ is piecewise constant (it has at most a finite number of jumps in each compact interval) and is continuous from the right. We assume that $f : \mathbb{R}^n \times \mathcal{I} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathcal{I} \to \mathbb{R}^p$ are continuous in $x \in \mathbb{R}^n$ for every fixed $i \in \mathcal{I}$ and write $f_i(\cdot) = f(\cdot, i)$ and $h_i(\cdot) = h(\cdot, i)$. We denote by $\mathcal{S}$ the set of all the switching signals. A (forward) solution of (1) corresponding to a switching signal $\sigma \in \mathcal{S}$ is a locally absolutely continuous function $x : [t_0(x), t_f(x)) \to \mathbb{R}^n$, with $0 \leq t_0(x) < t_f(x)$, such that $\dot{x}(t) = f(x(t), \sigma(t))$ for almost all $t \in [t_0(x), t_f(x))$. A solution $x : [t_0(x), t_f(x)) \to \mathbb{R}^n$ corresponding to $\sigma \in \mathcal{S}$ is maximal, if there does not exists a solution $\tilde{x} : [t_0(\tilde{x}), t_f(\tilde{x})) \to \mathbb{R}^n$ corresponding to σ such that $t_0(x) = t_0(\tilde{x})$, $t_f(\tilde{x}) > t_f(x)$ and $\tilde{x}(t) = x(t)$ for all $t \in [t_0(x), t_f(x))$. A pair (x, σ) is a trajectory of (1) if x is a maximal solution of (1) corresponding to $\sigma \in \mathcal{S}$. A trajectory (x, σ) is forward complete if $t_f(x) = +\infty$.

In many cases one is not interested in all the possible trajectories of (1) but in a subset $\mathcal{T}$ of them. For example, one can be interested only in those trajectories whose switched signals have some minimun (or maximum) dwell-time, or in those whose switchings verify some time/state-dependent rule. One of the main issues in the theory of switched systems is the analysis of the stability properties of a given family of trajectories $\mathcal{T}$, in particular of the global uniform stability (GUS) and of the global uniform asymptotic stability (GUAS) properties which we define below. Note that due to time invariance we can assume, without loss of generality, that the initial time of any trajectory (x, σ) of the family under analysis is $t_0(x) = 0$.

Definition 2.1: Let $\mathcal{T}$ be a family of trajectories (x, σ) of (1).

1) $\mathcal{T}$ is GUS if there exists $\alpha \in \mathcal{K}$ such that for any $(x, \sigma) \in \mathcal{T}$ we have $|x(t)| \leq \alpha(|x(s)|)$ for all $0 \leq s \leq t < t_f(x)$.
2) $\mathcal{T}$ is GUAS if there exists $\beta \in \mathcal{KL}$ such that for any $(x, \sigma) \in \mathcal{T}$ we have $|x(t)| \leq \beta(|x(s)|, t - s)$ for all $0 \leq s \leq t < t_f(x)$.

Remark 2.1:

1) The stability properties defined above can be equivalently formulated in the classical $\varepsilon - \delta$ form as is done, for example, in [16, Defn.1].
2) By using standard arguments of the theory of differential equations it can be proved that if $\mathcal{T}$ is GUS or GUAS then every trajectory in $\mathcal{T}$ is forward complete.

Remark 2.2: For non-switched time-invariant systems, i.e. systems of the form $\dot{x} = f(x)$, GUAS is equivalent to global asymptotic stability (GAS) that is, GUS plus the global attractiveness of the zero solution ($x(t) \to 0$ as $t \to \infty$ for every solution x). This equivalence is no longer true for families of trajectories of switched systems. For example, consider the switched system in $\mathbb{R}^2$ with two subsystems: $f_1(x) = [x_2 \ \ -x_1]'$ and $f_2(x) = -x$. Let $\mathcal{T}$ be the family of trajectories (x, σ) with $\sigma \in \mathcal{S}^*$, where $\mathcal{S}^*$ is the set of switching signals defined as follows: $\sigma \in \mathcal{S}^*$ if there exists $t^\sigma \geq 0$ such that $\sigma(t) = 1$ whenever $t \leq t^\sigma$ and $\sigma(t) = 2$ otherwise. It is easy to see that such a family $\mathcal{T}$ is GAS but not GUAS. The nonequivalence between GAS and GUAS makes it harder to prove the GUAS of a family $\mathcal{T}$, since one has to prove in addition that the attractiveness of the zero solution is uniform.

In many practical cases the family of trajectories $\mathcal{T}$ admits common or multiple weak Lyapunov functions, i.e. positive definite functions whose total derivative along the trajectories of $\mathcal{T}$ are nonpositive. In these cases the GUS of $\mathcal{T}$ can be straightforwardly asserted ([10]). Once the GUS property is established, the GUAS property of $\mathcal{T}$ can be obtained under additional conditions. A set of such additional conditions involves the use of an auxiliary output– which is often related to the total derivative of the weak Lyapunov functions used for establishing the GUS of $\mathcal{T}$– and the concept of weak zero-state detectability which we next recall (see [20]).

Definition 2.2: Let f and h be as in (1) and (2) respectively and let $\mathcal{T}$ be a family of trajectories of (1). The pair (h, f) is weak zero-state detectable (WZSD) with respect to (w.r.t.) $\mathcal{T}$ if for any $0 < \varepsilon \leq 1$ there are no sequences $\{(x_k, \sigma_k)\}$ in $\mathcal{T}$ and $\{t_k\} \subset \mathbb{R}$ such that for each $k \in \mathbb{N}$ the following hold:

- $k \leq t_k < t_f(x_k) - k$;
- $\varepsilon \leq |x_k(t_k + t)| \leq 1/\varepsilon$ for all $t \in [-k, k]$;
- For almost all $t \in \mathbb{R}$

$$\lim_{k \to \infty} h(x_k(t_k + t), \sigma_k(t_k + t)) = 0.$$

Assumption 1 below plays an important role in checking GUAS of (1) with the help of the output (2) as discussed in [16].

Assumption 1: For any $0 < \varepsilon \leq 1$ and any $\mu > 0$ there exists $M = M(\varepsilon, \mu) > 0$ such that for every (x, σ) in $\mathcal{T}$ and every $0 \le s < t < t_f(x)$ such that for all $\tau \in [s, t]$ $\varepsilon \le |x(\tau)| \le \frac{1}{\varepsilon}$, we have

$$\int_s^t |h(x(\tau), \sigma(\tau))|^2 \, d\tau \le M + \mu(t - s). \tag{3}$$

The following result follows from [16, Thm.1] and the fact that the WZSD of a pair (h, f) w.r.t. $\mathcal{T}$ implies that (h, f) is output-persistently exciting w.r.t. $\mathcal{T}$ (see [20, Lem.1]).

Theorem 2.1: Let $\mathcal{T}$ be a GUS family of trajectories of (1). Suppose that Assumption 1 holds and that the pair (h, f) WZSD w.r.t. $\mathcal{T}$. Then $\mathcal{T}$ is GUAS.

In the applications of Theorem 2.1 the more difficult task is to check the WZSD of (h, f) w.r.t. $\mathcal{T}$. Although some results for checking that property were given in [20] much remains to be investigated.

In many cases one knows, *a priori*, that the set of trajectories $\mathcal{T}$ under analysis satisfies some type of time-dependent constraint, for example its switching signals verify some dwell-time condition, and/or some state-dependent constraint, such as the invariance of $\mathcal{T}$ w.r.t. a closed covering $\chi = \{\chi_i\}_{i \in \mathcal{I}}$ of $\mathbb{R}^n$. We recall that $\mathcal{T}$ is invariant w.r.t. χ if for all $(x, \sigma) \in \mathcal{T}$, $x(t) \in \chi_{\sigma(t)}$ for all $t \in [0, t_f(x))$, or, equivalently, $\sigma(t) \in \mathcal{I}_{x(t)}$ for all $t \in [0, t_f(x))$, where, for a given $\xi \in \mathbb{R}^n$,

$$\mathcal{I}_\xi = \{i \in \mathcal{I} : \xi \in \chi_i\}. \tag{4}$$

This additional information may be useful for checking the WZSD of (h, f) w.r.t. $\mathcal{T}$.

The precedent discussion motivates one of the problems the paper deals with: the search of easier to check sufficient conditions for the WZSD of a pair (h, f) w.r.t. a family of trajectories $\mathcal{T}$, which take into account the time/state-dependent constraint the family must satisfy (see Section III). The other problem we consider is finding sufficient conditions for the GUAS of $\mathcal{T}$ based on weak Lyapunov functions and the sufficient conditions for the WZSD obtained (see Section IV).

III. WZSD OF SWITCHED SYSTEMS VIA WZSD OF REDUCED CONTROL SYSTEMS

In this section we study the WZSD of the pair (h, f) w.r.t. a given family of trajectories $\mathcal{T}$ of (1) through the limits of sequences like those appearing in Definition 2.2. For defining and characterizing those limits, we need to embed system (1)-(2) into the control-affine system with outputs

$$\dot{x} = F(x)u \tag{5}$$

$$y = H(x)u \tag{6}$$

where $F(\xi) := [f_1(\xi) \dots f_N(\xi)] \in \mathbb{R}^{n \times N}$ and $H(\xi) := [h_1(\xi) \dots h_N(\xi)] \in \mathbb{R}^{p \times N}$ for all $\xi \in \mathbb{R}^n$. We assume that the admissible controls of (5) belong to $\mathcal{U}$, the set of all the Lebesgue measurable functions $u : \mathbb{R} \to U$, where $U = \mathrm{co}(U^*)$, with $U^* = \{e_1, \dots, e_N\}$, and where $e_i \in \mathbb{R}^N$ denotes the i-th canonical vector of $\mathbb{R}^N$.

The embedding of (1) into (5) is performed by identifying the set $\mathcal{S}$ of all the switching signals with the set $\mathcal{U}_{pc}^*$ of all controls $u \in \mathcal{U}$ that take values in U^* and are piecewise constant and continuous from the right, by means of the bijection $\sigma \mapsto u_\sigma$, $u_\sigma(\cdot) = e_{\sigma(\cdot)}$. We note that the solutions of (1) corresponding to a switching signal σ are, respectively, the same as those of (5) which correspond to the control u_σ. It also holds that for any solution $x : [0, t_f(x)) \to \mathbb{R}^n$ of (1) corresponding to a switching signal σ,

$$h(x(t), \sigma(t)) = H(x(t))u_\sigma(t) \quad \forall t \in [0, t_f(x)). \tag{7}$$

So, in what follows we will identify every switching signal σ with the corresponding control u_σ, and write σ in place of u_σ without risk of confusion. In other words, depending on the context, σ will represent either a switching signal or the corresponding piecewise constant control u_σ. We will also identify the set $\mathcal{S}$ with $\mathcal{U}_{pc}^*$ and any family $\mathcal{T}$ of trajectories (x, σ) of (1) with the corresponding family of trajectories (x, u_σ) of (5).

A. Limiting trajectories

In order to define the concept of limiting trajectory of a family $\mathcal{T}$ of trajectories of (1), we need to consider the following notion of weak-convergence in $\mathcal{U}$: given a sequence $\{u_k\}$ in $\mathcal{U}$ and $u \in \mathcal{U}$, we say that $u_k \rightharpoonup u$ if for all $f \in L_N^1(\mathbb{R})$

$$\lim_{k \to \infty} \int_{-\infty}^{\infty} f(t)' u_k(t) \, dt = \int_{-\infty}^{\infty} f(t)' u(t) \, dt.$$

The next result about the sequential compactness of $\mathcal{U}$ will be used along the paper.

Lemma 3.1: For every sequence $\{u_k\}$ in $\mathcal{U}$ there exist $u \in \mathcal{U}$ and a subsequence $\{u_{k_l}\}$ such that $u_{k_l} \rightharpoonup u$.

Next we define the notion of limiting trajectory of a set $\mathcal{T}$ of trajectories of the switched system (1).

Definition 3.1: A pair $(\bar{x}, \bar{u})$, with $\bar{x} : \mathbb{R} \to \mathbb{R}^n$ and $\bar{u} \in \mathcal{U}$ is a limiting trajectory of $\mathcal{T}$ corresponding to the unbounded sequence $\gamma = \{t_k\}$ of positive real numbers if there exists a sequence $\{(x_k, \sigma_k)\}$ of trajectories in $\mathcal{T}$ and a compact set $K \subset \mathbb{R}^n$ such that:

1) For every k, $x_k(t) \in K$ for all $t \in [t_k - k, t_k + k] \subset [0, t_f(x_k))$, and $\{x_k(t_k + \cdot)\}$ converges to $\bar{x}$ uniformly on $[-T, T]$ for all $T > 0$; and
2) $\sigma_k(t_k + \cdot) \rightharpoonup \bar{u}$.

Note that for any limiting trajectory $(\bar{x}, \bar{u})$ of $\mathcal{T}$, $\bar{x}$ is a limiting solution of $\mathcal{T}$ in the sense given in [16, Defn. 2].

Lemma 3.2 below shows that if the trajectories of $\mathcal{T}$ are invariant w.r.t. a closed covering χ, then the limiting trajectories of $\mathcal{T}$ also satisfy a state-dependent constraint. For stating the type of constraint the limiting trajectories satisfy we define for $\xi \in \mathbb{R}^n$ the following set of control values:

$$U_\xi = \mathrm{co}\{e_i : i \in \mathcal{I}_\xi\}, \tag{8}$$

with $\mathcal{I}_\xi$ given by (4).

Lemma 3.2: Let $\mathcal{T}$ be a family of trajectories of (1) which is invariant w.r.t. a closed covering χ. Let $(\bar{x}, \bar{u})$ be a limiting trajectory of $\mathcal{T}$. Then $\bar{u}(t) \in U_{\bar{x}(t)}$ for almost all $t \in \mathbb{R}$.

B. Reduced control system

Now we introduce the notion of reduced control system for a family of trajectories $\mathcal{T}$ of (1)-(2). We will assume in the following that $\mathcal{T}$ is invariant w.r.t. some closed covering χ. This assumption does not imply any loss of generality since any family of trajectories $\mathcal{T}$ is always invariant w.r.t. the trivial covering $\chi = \{\chi_i\}_{i=1}^N$, where $\chi_i = \mathbb{R}^n$ for all i.

Let $\mathcal{S}_\mathcal{T} = \{\sigma \in \mathcal{S} : \exists\, x \text{ s.t. } (x, \sigma) \in \mathcal{T}\}$ and let $\mathcal{S}^*_\mathcal{T}$ be the set of all the controls $u \in \mathcal{U}$ for which there exist $\{t_k\} \in \mathbb{R}$, with $t_k \to +\infty$ and $\{\sigma_k\} \subset \mathcal{S}_\mathcal{T}$ so that $\sigma_k(t_k + \cdot) \rightharpoonup u$.

Then the reduced control system with outputs associated to the family of trajectories $\mathcal{T}$ of (1)-(2) is:

$$\Sigma : \begin{cases} \dot{x} = F(x)u \\ y = H(x)u = 0 \end{cases}, \quad u \in \mathcal{S}^*_\mathcal{T}, \quad u \in U_x \tag{9}$$

We say that (x, u) is a complete trajectory of Σ if $x : \mathbb{R} \to \mathbb{R}^n$ is locally absolutely continuous, $u \in \mathcal{S}^*_\mathcal{T}$ and, for almost all $t \in \mathbb{R}$, $u(t) \in U_{x(t)}$, $\dot{x}(t) = F(x(t))u(t)$ and $H(x(t))u(t) = 0$.

Remark 3.1: The reduced control system Σ is actually a differential-algebraic equation (DAE) with some additional constraints. For a DAE, existence and uniqueness of solutions is a nontrivial problem. However, this fact will not be an issue in our context, since we will not need to solve the equation but check that its solutions, if any, satisfy a certain condition which we specify later.

Remark 3.2: The reduced control system Σ incorporates the time/state-dependent constraints the family of trajectories $\mathcal{T}$ satisfies. On one hand the invariance of $\mathcal{T}$ w.r.t. the covering χ is taken into account through the restriction $u \in U_x$. On the other hand the restriction $\sigma \in \mathcal{S}^*_\mathcal{T}$ allows us to take into account the time-dependent constraints that the switching signals of the trajectories of $\mathcal{T}$ satisfy. For example, if the switching signals in $\mathcal{S}_\mathcal{T}$ have common average dwell-time $\tau_D > 0$ and chattering bound $N_0 \in \mathbb{N}$ (see [1]), then the controls in $\mathcal{S}^*_\mathcal{T}$ are controls in $\mathcal{U}^*_{pc}$ which have average dwell-time $\tau_D > 0$ and chattering bound $N_0 \in \mathbb{N}$. This assertion follows from the facts that $\mathcal{S}_\mathcal{T}$, as a subset of $\mathcal{U}^*_{pc}$, is a subset of $\mathcal{U}[\tau_D, N_0]$, the set of all controls in $\mathcal{U}^*_{pc}$ which have average dwell-time $\tau_D > 0$ and chattering bound $N_0 \in \mathbb{N}$, and that $\mathcal{U}[\tau_D, N_0]$ is invariant for time-translations and sequentially compact w.r.t. the weak convergence. That $\mathcal{U}[\tau_D, N_0]$ is sequentially compact w.r.t. the weak convergence follows from the fact that it is sequentially compact with respect to the almost everywhere convergence. More generally, it can be proved that if $\mathcal{S}_\mathcal{T}$ is contained in a set $\mathcal{V} \subset \mathcal{U}^*_{pc}$ which is invariant for time-translations and sequentially compact with respect to the almost everywhere convergence then $\mathcal{S}^*_\mathcal{T} \subset \mathcal{V}$ (see [11], [22] for examples of such a sets $\mathcal{V}$).

C. A criterion for WZSD of $\mathcal{T}$

Next we give a criterion for the WZSD of a pair (h, f) w.r.t. a family of trajectories $\mathcal{T}$ of (1)-(2). This criterion is formulated in terms of the WZSD of the reduced control system we introduced above.

Definition 3.2: The reduced control system Σ defined in (9) is WZSD if every complete trajectory (x, u) of Σ with x bounded satisfies: $\inf_{t\in\mathbb{R}} |x(t)| = 0$.

Theorem 3.1: Let $\mathcal{T}$ be a family of trajectories of (1)-(2) which is invariant w.r.t. a closed covering χ. Then the pair (h, f) is WZSD w.r.t. $\mathcal{T}$ if the reduced control system Σ in (9) is WZSD.

The complete proof of Theorem 3.1 is omitted due to space constraints. Nevertheless we give a sketch of it.

Sketch of the proof of Theorem 3.1. We prove it by contradiction. Suppose the pair (h, f) is not WZSD w.r.t. $\mathcal{T}$. Then there exist a sequence $\{t_k\}$ in $\mathbb{R}_{\geq 0}$ and a sequence $\{(x_k, \sigma_k)\}$ in $\mathcal{T}$ verifying the conditions in Definition 2.2. By using Arzela-Ascoli Theorem and Lemma 3.1, and passing to a subsequence if necessary, it can be proved the existence of a limiting trajectory $(\bar{x}, \bar{u})$ of $\mathcal{T}$ such that $x_k(t_k + \cdot) \to \bar{x}$ uniformly on $[-T, T]$ for all $T > 0$ and $\sigma_k(t_k + \cdot) \rightharpoonup \bar{u}$. In addition $\bar{x}$ is bounded and stays away from zero. On the other hand, it can be proved that the limiting trajectory $(\bar{x}, \bar{u})$ is a complete trajectory of the reduced control system Σ. Since Σ is assumed WZSD, $\inf_{t\in\mathbb{R}} |\bar{x}(t)| = 0$, which contradicts the fact that $\bar{x}$ stays away from zero. ∎

IV. A CRITERION FOR THE GUAS OF SWITCHED SYSTEMS

In this section we give a criterion for the GUAS of a family $\mathcal{T}$ of trajectories of (1). This criterion assumes the existence of weak multiple Lyapunov functions for a family $\mathcal{T}$ which is invariant w.r.t. a closed covering χ.

Assumption 2: There exists a function $V : \mathbb{R}^n \times \mathcal{I} \to \mathbb{R}_{\geq 0}$ such that:

1) There exist $\phi_1, \phi_2 \in \mathcal{K}_\infty$, such that for all $\xi \in \chi_i$ and all $i \in \mathcal{I}$
$$\phi_1(|\xi|) \leq V(\xi, i) \leq \phi_2(|\xi|); \tag{10}$$
2) for all $i \in \mathcal{I}$, $V_i(\cdot) = V(\cdot, i)$ is continuously differentiable and for all $\xi \in \chi_i$ and all $i \in \mathcal{I}$
$$\dot{V}_i(\xi) = \nabla V_i(\xi) f_i(\xi) \leq -\eta_i(\xi), \tag{11}$$
where $\eta_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a continuous function;
3) for every $(x, \sigma) \in \mathcal{T}$ and every $0 \leq t < s < t_f(x)$ such that $\sigma(t) = \sigma(s) = i$ we have that
$$V_i(x(t)) \geq V_i(x(s)).$$

The following criterion is a consequence of Theorems 2.1 and 3.1 and well-known results of the stability theory of switched systems.

Theorem 4.1: Let $\mathcal{T}$ be a family of trajectories of (1) which is invariant w.r.t. a closed covering χ. Suppose that

Assumption 2 holds and let h in (2) be defined by $h(\xi, i) = \sqrt{\eta_i(\xi)}$ for all $\xi \in \mathbb{R}^n$ and all $i \in \mathcal{I}$.

Then $\mathcal{T}$ is GUAS if the reduced control system Σ defined in (9) is WZSD.

Sketch of the proof. Assumption 2 implies that $\mathcal{T}$ is GUS and that Assumption 1 holds for the output map h. The fact that Σ is WZSD implies that (h, f) is WZSD w.r.t. $\mathcal{T}$ according to Theorem 3.1. Then, that $\mathcal{T}$ is GUAS follows from Theorem 2.1. ∎

Remark 4.1: As a difference with most GUAS results based on multiple weak Lyapunov functions, Theorem 4.1 does not involve any dwell-time assumption on the switching signals of the trajectories of the family $\mathcal{T}$. In fact, it can be used for proving the GUAS of families of trajectories whose switching signals do not satisfy any dwell-time condition (see Example 4.1). Even more, from Theorem 4.1 we will derive a criterion for GUAS under arbitrary switchings (see Corollary 4.1 below).

A. A criterion for GUAS under arbitrary switching

We say that (1) is GUAS under arbitrary switching if the family $\mathcal{T}^*$ of all the trajectories of (1) is GUAS. A criterion for GUAS under arbitrary switching can be easily derived from Theorem 4.1 assuming the existence of a common weak Lyapunov function.

Assumption 3: There exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that:

1) There exist $\phi_1, \phi_2 \in \mathcal{K}_\infty$, such that for all $\xi \in \mathbb{R}^n$ for all $\xi \in \mathbb{R}^n$
$$\phi_1(|\xi|) \leq V(\xi) \leq \phi_2(|\xi|); \quad (12)$$
2) for each $i \in \mathcal{I}$ there exists a continuous function $\eta_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that
$$\dot{V}_i(\xi) = \nabla V(\xi) f_i(\xi) \leq -\eta_i(\xi). \quad (13)$$

Corollary 4.1: Let Assumption 3 hold for system (1). Let h in (2) be defined by $h(\xi, i) = \sqrt{\eta_i(\xi)}$ for all $\xi \in \mathbb{R}^n$ and all $i \in \mathcal{I}$. If the reduced control system
$$\Sigma : \begin{cases} \dot{x} = F(x)u \\ y = H(x)u = 0 \end{cases}, \quad u \in \mathcal{U} \quad (14)$$
is WZSD, then (1) is GUAS under arbitrary switching.

The following criterion for GUAS under arbitrary switching, which was recently obtained in [20, Thm. 4], is a straightforward consequence of Corollary 4.1.

Corollary 4.2: Suppose that Assumption 3 holds for system (1). Suppose in addition that there exist continuous functions f_c and $g_1, \dots, g_N$ such that for all $i \in \mathcal{I}$

1) $f_i = f_c + g_i$;
2) $g_i(\xi) = 0$ when $\eta_i(\xi) = 0$.

Let $h^* = \prod_{i=1}^N \eta_i$. Then (1) is GUAS under arbitrary switching if the following reduced system is WZSD
$$\Sigma^* : \begin{cases} \dot{x} = f_c(x) \\ y = h^*(x) = 0 \end{cases}.$$

B. Example

In the following example we apply Theorem 4.1 for asserting the GUAS of a family of trajectories which satisfy a state-dependent constraint but whose switching signals do not verify any dwell-time condition.

Example 4.1: Consider the switched system (1) in $\mathbb{R}^2$ with three modes, i.e. $\mathcal{I} = \{1, 2, 3\}$, given by
$$f_1(\xi) = \begin{bmatrix} \xi_2 \\ -\xi_1 - \xi_2 \end{bmatrix}, \ f_2(\xi) = \begin{bmatrix} -\xi_1^{\frac{1}{3}} - \xi_2 \\ \xi_1 \end{bmatrix}$$
and
$$f_3(\xi) = \begin{bmatrix} 3\xi_1 + 5\xi_2 \\ -5\xi_1 - 3\xi_2 \end{bmatrix}.$$
We will show that the family $\mathcal{T}$ of all the forward complete trajectories of (1) which is invariant w.r.t. the covering $\chi = \{\chi_1\}_{i=1}^3$, where $\chi_1 = \chi_2 = \{\xi \in \mathbb{R}^2 : \xi_1 \geq 0\}$ and $\chi_3 = \{\xi \in \mathbb{R}^2 : \xi_1 \leq 0\}$ is GUAS. Let $V : \mathbb{R}^2 \times \mathcal{I} \to \mathbb{R}$ be defined via $V_1(\xi) = V_2(\xi) = 5\xi_1^2 + 5\xi_2^2$ and $V_3(\xi) = 5\xi_1^2 + 6\xi_1\xi_2 + 5\xi_2^2$. It is clear that V satisfies Assumption 2 with $\eta_1(\xi) = 10\xi_2^2$, $\eta_2(\xi) = 10\xi_1^{\frac{4}{3}}$ and $\eta_3(\xi) = 0$. If we consider the output (2) with h defined by $h_i = \sqrt{\eta_i}$ for $i \in \mathcal{I}$, we have that the reduced control system defined by (9) is
$$\Sigma : \begin{cases} \dot{x} = \sum_{i=1}^3 u_i f_i(x) \\ y = \sum_{i=1}^3 u_i h_i(x) = 0 \end{cases}, \quad u \in \mathcal{S}_{\mathcal{T}}^*, \ u \in U_x$$
where $U_\xi = \{e_3\}$ if $\xi_1 < 0$, $U_\xi = \text{co}\{e_1, e_2\}$ if $\xi_1 > 0$ and $U_\xi = \text{co}\{e_1, e_2, e_3\}$ if $\xi_1 = 0$.

Let (x, u) be a complete trajectory of Σ such that $x(t) \neq 0$ for all $t \in \mathbb{R}$. We consider two possible cases.

Case I. There exits $\bar{t}$ such that $x(\bar{t}) \in \chi_1^o = \{\xi : \xi_1 > 0\}$. In this case, by continuity, $x(t) \in \chi_1^o$ on some interval $[\bar{t}, T]$, with $T > \bar{t}$, and hence $u(t) \in \text{co}\{e_1, e_2\}$ for almost all $t \in [\bar{t}, T]$. Then, for almost all $t \in [\bar{t}, T]$
$$\dot{x}(t) = u_1(t) f_1(x(t)) + u_2(t) f_2(x(t)),$$
$u_1(t)|x_2(t)| + u_2(t)|x_1(t)|^{\frac{2}{3}} = 0$, $u_1(t) + u_2(t) = 1$ and $u_i(t) \geq 0$ for $i = 1, 2$. Since $x_1(t) > 0$ for all $t \in [\bar{t}, T]$, $u_2(t) = 0$ for almost all $t \in [\bar{t}, T]$. Therefore, for almost all $t \in [\bar{t}, T]$ we have that
$$\dot{x}(t) = f_1(x(t)) \quad \text{and} \quad x_2(t) = 0.$$
Since f_1 and x are continuous, the equalities above hold for every $t \in [\bar{t}, T]$. This implies that $x_1(t) = 0$ for all $t \in [\bar{t}, T]$ which is a contradiction. So, case I is not possible.

Case II. $x(t) \in \chi_3$ for all $t \in \mathbb{R}$. Since while x remains in $\chi_3^o = \{\xi : \xi_1 < 0\}$, x is a solution of the linear differential equation $\dot{x} = f_3(x)$, and the orbits of that system are clockwise ellipses, there exists $\bar{t} \in \mathbb{R}$ such that $x(\bar{t}) \in \partial\chi_3 = \{\xi : \xi_1 = 0\}$. Suppose $x(t) \in \partial\chi_3$ for all $t \geq \bar{t}$. Since $x_2(t) \neq 0$ for all $t \geq \bar{t}$ because we suppose that $x(t) \neq 0$ for all $t \in \mathbb{R}$, it follows that $u_1(t) = 0$ for almost all $t \in [\bar{t}, \infty)$. In consequence $\dot{x}(t) = u_2(t) f_2(x(t)) + u_3(t) f_3(x(t))$ for almost all $t \in [\bar{t}, \infty)$. Since $x_1(t) = 0$ for all $t \in [\bar{t}, \infty)$, it follows that $0 = \dot{x}_1(t) = [-u_2(t) + 5u_3(t)]x_2(t)$ must hold almost everywhere on $[\bar{t}, \infty)$. So $u_2(t) = 5/6$ and $u_3(t) = 1/6$ for almost all $t \in [\bar{t}, \infty)$. Therefore, $\dot{x}_2(t) = -3/6x_2(t)$ for almost all $t \geq \bar{t}$, and $x_2(t) = x_2(\bar{t})e^{-3/6(t-\bar{t})}$. The latter shows that $x(t) \to 0$ as $t \to +\infty$ and $\inf_{t \in \mathbb{R}} |x(t)| = 0$.

Suppose that x does not remain in $\partial\chi_3$ for all $t \geq \bar{t}$. We will consider two cases.
i) $x_2(\bar{t}) > 0$. There exists $t' \geq \bar{t}$ such that $x(t) \in \partial\chi_3$ for all $t \in [\bar{t}, t']$ and a sequence $\{t_k\}$ such that $t_k \searrow t'$ and $x(t_k) \in \chi_3^o$. Note that by continuity and the fact that $x(t) \neq 0$ for all $t \in \mathbb{R}$, $x_2(t') > 0$. Since $t_k \searrow t'$ and $x(t_k) \in \chi_3^0$, for each k there exists $\delta_k > 0$, with $\delta_k \to 0$ such that $x(t_k - \delta_k) \in \partial\chi_3$ and $x(t) \in \chi_3^o$ for all $t \in (t_k - \delta_k, t_k]$. Then $x_2(t_k - \delta_k) < 0$ for all k, since x is a solution of $\dot{x} = f_3(x)$ on $(t_k - \delta_k, t_k]$ and the nontrivial solutions of this equation are clockwise ellipses. So $x_2(t') = \lim_{k\to\infty} x_2(t_k - \delta_k) \leq 0$, which is a contradiction. So, case i) is impossible.

ii) $x_2(\bar{t}) < 0$. Since x leaves $\partial\chi_3$, there exists $t' > \bar{t}$ such that $x(t') \in \chi_3^o$. Then, by using the facts that while x remains in $\chi_3^o = \{\xi : \xi_1 < 0\}$, x is a solution of the linear differential equation $\dot{x} = f_3(x)$, and the solutions of that equations are clockwise ellipses, there exists $t'' > t'$ such that $x(t'') \in \partial\chi_3$ and $x_2(t'') > 0$. Then, reasoning as in case i), we conclude that case ii) is also impossible.

We have then proved that for every complete trajectory (x, u) of the reduced control system Σ it holds that $\inf_{t\in\mathbb{R}} |x(t)| = 0$. So Σ is WZSD and the $\mathcal{T}$ is GUAS.

Remark 4.2: To establish the GUAS of the family of trajectories $\mathcal{T}$ considered in Example 4.1 is challenging. On one hand, it seems difficult to find a family of strong Lyapunov functions V_i that satisfy the conditions in Assumption 2, which are standard conditions for establishing GUAS by means of multiple strict Lyapunov functions. On the other hand, the existing results based on multiple weak Lyapunov functions, namely extensions of LaSalle's invariance principle or of the Krasovskii-LaSalle theorem, cannot be applied to this example since they assume the trajectories of switched system satisfy some kind of dwell-time constraint, and the trajectories of $\mathcal{T}$ do not satisfy any of them. Neither the results in [20], which do not make any dwell-time assumption, can be applied to this example since the system does not have a CZOS. The only results we are aware of, that could be applied to our example are those in [18], but they require to find an additional Lyapunov-like function W verifying condition (H2) in [18] and such that the pair $((h, \dot{W}), f)$ be output-persistently excited (or WZSD) w.r.t. $\mathcal{T}$, that in this case seems a nontrivial task due to the facts that f_3 is stable, but not asymptotically.

V. CONCLUSIONS

In this paper we have obtained a sufficient condition for the WZSD of time-invariant switched systems with outputs and time/state-dependent constraints. This condition is formulated in terms of the WZSD of a reduced control system with outputs, being this condition easier to check than the WZSD of the switched system itself. We have also derived a criterion for the GUAS of that kind of systems. This criterion is based on the existence of multiple weak Lyapunov functions and the WZSD of a reduced control system with an output related to the derivatives of the Lyapunov functions, and does not require any dwell-time assumption. As a byproduct of this result a criterion for GUAS under arbitrary switching was obtained. Finally, we have given an example for illustrating the use of the GUAS criterion.

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