On Bounding a Nonlinear System with a Monotone Positive System

Hernan Haimovich
Centro Internacional Franco-Argentino de
Ciencias de la Información y de Sistemas (CIFASIS)
CONICET-UNR, 2000 Rosario, Argentina.
haimovich@cifasis-conicet.gov.ar

José Luis Mancilla-Aguilar
Centro de Sistemas y Control
Instituto Tecnológico de Buenos Aires
Buenos Aires, Argentina
jmancill@itba.edu.ar

***Abstract*—How to bound the state vector trajectory of a nonlinear system in a way so that the obtained bound be of practical value is an open problem. If some norm is employed for bounding the state vector trajectory, then this norm should be carefully selected and the state vector components suitably scaled. In addition, practical applications usually require separate bounds on every state variable. Bearing this context in mind, we develop a novel componentwise bounding procedure applicable to both real and complex nonlinear systems with additive disturbances. A bound on the magnitude of the evolution of each state variable is obtained by computing a single trajectory of a well-specified "bounding" system constructed from the original system equations and the available disturbance bounds. The bounding system is shown to have highly desirable properties, such as being monotone and positive. We provide preliminary results establishing that key stability features are preserved by the bounding system for systems in triangular form.**

***Index Terms*—Componentwise bounds, monotone systems, positive systems, nonlinear systems.**

I. Introduction

The obtention of bounds on the trajectory of a system is a central issue in stability analysis. For example, the now standard definition of local Lyapunov stability of an equilibrium point requires that the norm of the (forward) state trajectory be bounded by a class-$\mathcal{K}$ function of the norm of the initial state (see, e.g. Chapter 4 of [10]). From a qualitative point of view, the specific norm employed for bounding the state vector is usually irrelevant, provided the norm is taken from a class of equivalent ones. By contrast, when practical issues such as estimating basins of attraction or the size of some ultimate bound are considered, then the question of what specific norm to use may be crucial in order to get useful estimates.

When the state equations arise from modelling a real system, it is often the case that each component of the state vector represents some physical quantity whose evolution should be contained within some given set. In this context, the obtention of bounds on each component of the state vector becomes especially relevant. If a bound on some norm of the state is to be of any use, then this norm should be carefully selected and/or the state equations should be suitably scaled.

This paper addresses the problem of obtaining bounds on the magnitude of each component of the state vector, i.e. componentwise magnitude bounds. Nonlinear systems with additive disturbances are considered, where a bound on the magnitude of each component of the disturbance vector is known. In the case when the system is input-to-state stable (ISS) [18], [19], regarding the disturbance vector as the input, then the ISS property provides a bound on the norm of the state vector depending on both the norm of the initial state and the maximum norm of the disturbance. Once a bound for the norm of the state is obtained, then componentwise bounds can be easily computed. However, for these bounds to be of practical value, the comparison functions characterizing ISS should be tight, and suitable state and disturbance norms have to be selected. The computation of tight comparison functions is a highly non-trivial task [9], [14], [20]–[22], even when the state and disturbance norms are given.

The obtention of componentwise bounds for linear time-invariant (LTI) systems with disturbances has been addressed in [13] for the case of constant (and also for some specific state-dependent) disturbance bounds. The main method in [13] has been tailored to sampled-data systems with quantization [4] and extended to switched systems [6], [7]. Also, discrete-time [8], [13] and probabilistic [11], [12] extensions are available. All of these previous works are essentially based on the analysis of a perturbed LTI system. The essential procedure consists in (a) finding a linear coordinate change under which the LTI system's A-matrix becomes simpler, e.g. diagonal or triangular; (b) bounding the magnitude of each component of the transformed system state; (c) linearly transforming back into the original coordinates. One key feature of this procedure is that the componentwise bounding step (b), which requires the transformed A-matrix to be "simpler", is guaranteed to produce a convergent bound whenever the nominal (i.e. undisturbed) system is asymptotically (hence globally exponentially) stable. The bounding step of the procedure, namely (b), is applicable only to an LTI system, with no obvious way to extend it to nonlinear systems.

In this context, the main contribution of the current paper is to provide preliminary results in order to develop a novel componentwise bounding procedure applicable to nonlinear systems and able to preserve key stability features when the nonlinear system is in some "simple" form (triangular). The componentwise bounds are obtained by computing a single trajectory of a (nonlinear) monotone positive system [1], [17]. Useful properties of monotone systems make this specific bounding procedure highly desirable [2]. As opposed to the LTI case where local stability notions are equivalent to global ones, and where trajectories are always defined for all times, the situation for nonlinear systems is substantially more delicate. Therefore, the developed bounding procedure exhibits key differences with respect to the previously existing LTI procedure. However, when applied to an LTI system, the bound obtained is shown to coincide with that obtained by the existing procedure in [5], [6].

The remainder of this paper proceeds as follows. This section ends with an overview of the notation employed. In Section II, we state the problem and define the bounding system. In Section III, we provide the main results, establishing properties of the bounding system and proof that its solution constitutes a componentwise bound on the magnitude of the state trajectory. In Section IV, we show that local exponential stability of the origin is maintained by the bounding system if the original system is triangular. Numerical examples illustrating the bounding procedure are given in Section V and conclusions are drawn in Section VI.

Notation: $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{C}$ denote the sets of natural, real, nonnegative real and complex numbers, respectively. For $n \in \mathbb{N}$, $\underline{n} := \{1, 2, \ldots, n\}$. $|M|$ denotes the *elementwise* magnitude of a matrix or vector M. The (i, k)-th entry of a matrix M is denoted by $M_{i,k}$. If $X, Y \in \mathbb{R}^{n\times m}$, the expression '$X \preceq Y (X \prec Y)$' denotes the set of componentwise inequalities $X_{i,k} \leq Y_{i,k} (X_{i,k} < Y_{i,k})$, $i \in \underline{n}$, $k \in \underline{m}$, between the entries of X and Y, and similarly for $X \succeq Y$. The imaginary unit is denoted j ($j^2 = -1$) and $\mathbf{1}$ denotes a vector all of whose components equal 1. A matrix $M \in \mathbb{R}^{n\times n}$ is *Metzler* if $M_{i,k} \geq 0$ for all $i \neq k$. For $x \in \mathbb{R}^n$ or $\mathbb{C}^n$, $\|x\|$ denotes its supremum (infinity) norm. For $x : I \subset \mathbb{R} \to \mathbb{R}^n$, where I is an open interval, $D^+x(t)$ denotes the upper-right Dini derivative:

$$D^+x(t) = \limsup_{h\to 0^+} \frac{x(t + h) - x(t)}{h}.$$

II. Preliminaries

As mentioned in Section I, part of the motivation for the componentwise bounding procedure in this paper arises in extending to nonlinear systems a previously existing procedure only applicable to LTI systems and requiring, in essence, an A-matrix of diagonal or triangular form. If the original A-matrix is not in such a form, then a linear change of coordinates in the state variables induces a similarity transformation on the A-matrix. For the transformed A-matrix to be diagonal or triangular, complex-valued entries may be needed. Taking this situation into account, we develop the componentwise bounding procedure for both complex and real systems.

A. Problem Statement

Consider the real or complex dynamic system with additive disturbances, given by[1]

$$\dot{z} = f(z) + w, \quad z(0) = z_0, \tag{1}$$

[1]Future work is aimed at extending the procedure to systems of more general forms, such as $\dot{z} = f(z) + g(z)w$, or $\dot{z} = f(z, w)$.

with $f : \mathbb{F}^n \to \mathbb{F}^n$ and $\mathbb{F} = \mathbb{R}$ in the real case or $\mathbb{F} = \mathbb{C}$ in the complex case. We assume that f is continuously differentiable everywhere (holomorphic in the complex case) and $f(0) = 0$. The disturbance vector w satisfies

$$|w(t)| \preceq \mathbf{w}, \quad \forall t \geq 0, \tag{2}$$

for some $\mathbf{w} \in \mathbb{R}^n_{\geq 0}$. The problem addressed is to obtain componentwise bounds on the magnitude vector $\rho = |z|$. This is achieved by designing a monotone positive (real) system $\dot{x} = \tilde{f}(x, \mathbf{w})$, so that $\rho(t) \preceq x(t)$ for all t for which x exists.

B. The Bounding System

To define the bounding system, we need the following definitions. For a matrix $N \in \mathbb{C}^{n\times n}$, define $\mathcal{M}(N) \in \mathbb{R}^{n\times n}$ as the matrix whose entries satisfy

$$[\mathcal{M}(N)]_{i,k} = \begin{cases} \mathbb{R}\mathrm{e}\{N_{i,k}\} & \text{if } i = k, \\ |N_{i,k}| & \text{if } i \neq k, \end{cases} \tag{3}$$

for all $i, k \in \underline{n}$. Also, $\mathcal{M}_i(N)$ will be used to denote the i-th row of $\mathcal{M}(N)$. Note that $\mathcal{M}(N)$ is Metzler for every $N \in \mathbb{C}^{n\times n}$. Given the continuously differentiable vector field f in (1), define the real vector field $\mathcal{M}[f] : \mathbb{R}^n \to \mathbb{R}^n$ via

$$\mathcal{M}[f]_i(x) := \int_0^1 \max_{y\in\mathcal{V}_i(\sigma|x|)} \left[\mathcal{M}_i\left(\frac{\partial f}{\partial z}(y)\right) |x| \right] d\sigma, \tag{4}$$

$$\mathcal{V}_i(p) := \{z \in \mathbb{F}^n : |z| \preceq p, |z_i| = p_i\}, \tag{5}$$

where $\mathcal{M}[f]_i$ denotes the i-th component of $\mathcal{M}[f]$ and the above definitions are valid for all $i \in \underline{n}$. Note that $z \in \mathcal{V}_i(|z|)$ for all $z \in \mathbb{F}^n$ and all $i \in \underline{n}$, $\mathcal{V}_i(0) = \{0\}$, and $\mathcal{M}[f](0) = 0$. In the next section, we will prove that if z is a solution to (1), then $\rho = |z|$ will satisfy $\rho(t) \preceq x(t)$, where x is a solution to the system

$$\dot{x} = \mathcal{M}[f](x) + \mathbf{w}, \quad x(0) \succeq \rho(0) = |z(0)|. \tag{6}$$

III. The Bounding Procedure

In order to establish that the solutions of (6) constitute componentwise magnitude bounds for the solutions of (1), we need several auxiliary results. Some of these results are interesting in their own right. In Section III-A, we prove some properties of the vector field $\mathcal{M}[f]$. In Section III-B, we derive a multivariable comparison lemma suited specifically to the current purpose. In Section III-C, we prove the main result of the section employing the results in Sections III-A and III-B.

A. Properties of $\mathcal{M}[f]$

Our first result is needed to ensure uniqueness of solutions of the bounding system (6).

Lemma 3.1: $\mathcal{M}[f]$ is locally Lipschitz continuous in either of these cases:

i) $\mathbb{F} = \mathbb{C}$, or
ii) $\mathbb{F} = \mathbb{R}$ and f has locally Lipschitz partial derivatives.

The proof of Lemma 3.1 is based on results for marginal functions (see for example [3, Prop. 2.10]) and is omitted due to lack of space. The requirement of local Lipschitzianity of the partial derivatives of f can be dropped in the complex case because a continuously differentiable holomorphic function has continuous partial derivatives of all orders.

Definition 3.1 (Adapted from [17]): Let $g : \mathbb{R}_{\geq 0}\times\mathbb{R}^n \to \mathbb{R}^n$. We say that g is of type K in some set $D \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ if whenever $(t, x), (t, y) \in D$, $x \preceq y$ and $x_i = y_i$ for some $i \in \underline{n}$, then $g_i(t, x) \leq g_i(t, y)$. We say that $\bar{g} : \mathbb{R}^n \to \mathbb{R}^n$ is of type K in $\bar{D} \subset \mathbb{R}^n$ if g defined via $g(t, x) := \bar{g}(x)$ is of type K in $\mathbb{R}_{\geq 0} \times \bar{D}$.

Remark 1: The type K condition is the main condition that gives rise to a monotone system, and is also referred to as the Kamke condition [17]. ◦

Lemma 3.2: The vector field $\mathcal{M}[f]$ is of type K in $\mathbb{R}^n_{\geq 0}$.

Proof: Let $0 \preceq x \preceq w$ so that $x_i = w_i$ for some $i \in \underline{n}$. From (5), then $\mathcal{V}_i(\sigma x) \subset \mathcal{V}_i(\sigma w)$ for all $\sigma \geq 0$. From (3) and (4), we have

$$\int_0^1 \max_{y\in\mathcal{V}_i(\sigma x)} \left[\mathrm{Re}\left\{\frac{\partial f_i}{\partial z_i}(y)\right\} x_i + \sum_{\substack{k=1\\k\neq i}}^{n} \left|\frac{\partial f_i}{\partial z_k}(y)\right| x_k \right] d\sigma$$
$$\leq \int_0^1 \max_{y\in\mathcal{V}_i(\sigma w)} \left[\mathrm{Re}\left\{\frac{\partial f_i}{\partial z_i}(y)\right\} w_i + \sum_{\substack{k=1\\k\neq i}}^{n} \left|\frac{\partial f_i}{\partial z_k}(y)\right| w_k \right] d\sigma$$

and hence $\mathcal{M}[f]_i(x) \leq \mathcal{M}[f]_i(w)$. ■

Remark 2: It is interesting to see what $\mathcal{M}[f]$ looks like for a linear system, i.e. $f(z) = Az$, with $A \in \mathbb{F}^{n\times n}$. In this case, it follows straightforwardly from (3) and (4), that $\mathcal{M}[f](x) = \mathcal{M}(A)|x|$, where $\mathcal{M}(A)$ is Metzler. The system $\dot{x} = \mathcal{M}(A)x$ is essentially the bounding system employed in [5], [6]. ◦

B. Multivariable comparison Lemma

The following result extends the well-known comparison Lemma (cf. Lemma 3.4 in [10]) to a multivariable setting. The main difference with existing multivariable comparison results such as Theorem 3.1 of [15] is that we do not require that solutions take values in an open set, and we employ upper-right Dini derivatives instead of standard derivatives.

Lemma 3.3 (Multivariable comparison Lemma): Let $D \subset \mathbb{R}^n$, $D = \prod_{k=1}^n [a_k, b_k)$ with $-\infty < a_k < b_k \leq \infty$ for all $k \in \underline{n}$. Let $g : \mathbb{R}_{\geq 0} \times D \to \mathbb{R}^n$ be such that $g(t, x)$ is continuous in (t, x) and locally Lipschitz in x (uniformly over t in compact sets), and suppose that g is of type K in $\mathbb{R}_{\geq 0} \times D$. Consider the initial value problem

$$\dot{x} = g(t, x), \quad x(t_0) = x_0, \quad t_0 \geq 0,$$

and let I_x be the maximal (forward) interval of existence of the solution x. Let $y(t)$ be continuous and satisfy

$$D^+y(t) \preceq g(t, y(t)), \quad y(t_0) \preceq x_0, \tag{7}$$

and $y(t) \in D$ for all $t \in I_y = [t_0, T_y)$, for some $T_y > t_0$. Then, $y(t) \preceq x(t)$ for all $t \in I_x \cap I_y$.

Sketch of the proof: The maximal interval of existence of x is a nonempty interval of the form $I_x = [t_0, T_x)$ or $[t_0, T_x]$ with $T_x \geq t_0$. If $T_x = t_0$ then the thesis trivially holds. Assume that $T_x > t_0$. For every $m \in \mathbb{N}$, consider the following auxiliary initial value problems

$$\dot{x}^m = g(t, x^m) + \frac{r}{m}\mathbf{1}, \quad x^m(t_0) = x_0 + \frac{r}{m}\mathbf{1},$$

where $\mathbf{1} = [1, 1, \ldots, 1]'$ is a vector all of whose components equal 1 and $r > 0$ is such that $x_0 + r\mathbf{1} \in D$. Since $x_0 + \frac{r}{m}\mathbf{1}$ lies in the interior of D, the maximal interval of existence of x^m is of the form $I_m = [t_0, T_m)$ or $I_m = [t_0, T_m]$ with $T_m > t_0$. Note that if $I_m = [t_0, T_m)$ then there exists $i \in \underline{n}$ such that $x_i^m(t) \to b_i$ as $t \to T_m^-$ and if $I_m = [t_0, T_m]$ then there exists $i \in \underline{n}$ so that $x_i^m(T_m) = a_i$. The facts that g is of type K and that (7) holds on $I_{y,m} = I_y \cap I_m$, imply Claim 1.

Claim 1: $y(t) \preceq x^m(t)$ for all $t \in I_{y,m} = I_y \cap I_m$. ◦

Claim 2: Let $t_0 < t_1 < T_x$. Then, there exists $m_0 \in \mathbb{N}$ such that $t_1 < T_m$ for all $m \geq m_0$, and $x^m \to x$ uniformly on $[t_0, t_1]$.

Proof of Claim 2: We first prove that for all $m \in \mathbb{N}$,

$$x_k(t) < x_k^m(t) \quad \forall t \in [t_0, t_1] \cap I_m, \forall k \in \underline{n}. \tag{8}$$

For a contradiction, suppose there exist $m \in \mathbb{N}$, $\tau \in [t_0, t_1] \cap I_m$ and $k \in \underline{n}$ such that $x_k(\tau) \geq x_k^m(\tau)$. Since $x_k(t_0) < x_k^m(t_0)$, and x and x^m are continuous, then there exists $\tau' \in (t_0, \tau]$ such that $x_k(\tau') = x_k^m(\tau')$. For every $i \in \underline{n}$, define

$$s_i := \inf\{t \in [t_0, t_1] \cap I_m : x_i(t) = x_i^m(t)\},$$
$$\underline{s} := \min_{i\in\underline{n}} s_i, \qquad I := \{i \in \underline{n} : s_i = \underline{s}\}.$$

The fact that $x_k(t_0) < x_k^m(t_0)$ for all $k \in \underline{n}$, the continuity of x and x^m and the definition of $\underline{s}$ imply that $t_0 < \underline{s} \leq \tau'$ and that $x_i(\underline{s}) = x_i^m(\underline{s})$ for all $i \in I$, $x(\underline{s}) \preceq x^m(\underline{s})$ and $x_k(s) < x_k^m(s)$ for all $s \in [t_0, \underline{s})$ and all $k \in \underline{n}$. Then, for $i \in I$, since $\dot{x}_i^m(\underline{s}) = g_i(\underline{s}, x^m(\underline{s})) + \frac{r}{m}$ and $\dot{x}_i(\underline{s}) = g_i(\underline{s}, x^m(\underline{s}))$, it follows that

$$g_i(\underline{s}, x^m(\underline{s})) + \frac{r}{m} = \lim_{s\to\underline{s}^-} \frac{x_i^m(s) - x_i^m(\underline{s})}{s - \underline{s}}$$
$$\leq \lim_{s\to\underline{s}^-} \frac{x_i(s) - x_i(\underline{s})}{s - \underline{s}} = g_i(\underline{s}, x(\underline{s})).$$

Also, since g is of type K, then $g_i(\underline{s}, x(\underline{s})) \leq g_i(\underline{s}, x^m(\underline{s}))$. We have thus reached a contradiction.

Let $\mu > 0$ be such that the set $C = \{y \in \mathbb{R}^n : \exists t \in [t_0, t_1], x(t) \preceq y \preceq x(t)+\mu\mathbf{1}\} \subset D$. Note that C is a compact set and let L be a Lipschitz constant for g on $[t_0, t_1] \times C$, i.e. $L > 0$ satisfies

$$\|g(t, y) - g(t, \tilde{y})\| \leq L\|y - \tilde{y}\|, \quad \forall t \in [t_0, t_1], \forall y, \tilde{y} \in C.$$

Let $t^m = \sup\{t \in [t_0, t_1] : x^m(\tau) \in C, \ \forall t_0 \leq \tau \leq t\}$. The continuity of x and x_m, the fact that $x^m(t_0) = x(t_0) + \frac{r}{m}\mathbf{1}$ and the definition of C, ensure the existence of m_0^* such that $t_0 < t^m \leq t_1$ for all $m \geq m_0^*$. Applying Gronwall's Lemma, it can be proved that for all $t \in [t_0, t^m)$

$$\|x^m(t) - x(t)\| \leq \frac{r}{m}\|\mathbf{1}\|(1 + t_1 - t_0)e^{L(t_1-t_0)}. \tag{9}$$

Let $m_0 \geq m_0^*$ be any natural number such that

$$\frac{r}{m_0}\|\mathbf{1}\|(1 + t_1 - t_0)e^{L(t_1-t_0)} \leq \frac{\mu}{2}. \tag{10}$$

By using (8), (9) and (10), and taking into account the definition of C it follows that for all $m \geq m_0$, $x_m(t^m)$ lies in the interior of C and therefore $t^m = t_1$. Then, the uniform convergence of x^m to x on $[t_0, t_1]$ follows from (9). ◦

We finish the proof of the lemma as follows. Suppose that $T_y \leq T_x$. Then, $I_x \cap I_y = [t_0, T_y)$ and let $t \in [t_0, T_y)$. From Claim 2, x^m is defined on $[t_0, t]$ for m large enough and $x_m(t) \to x(t)$. Also, from Claims 1 and 2, $y_k(t) \leq x_k^m(t)$ for all k if m is large enough. By letting m go to ∞ we obtain $y_k(t) \leq x_k(t)$ for all $k \in \underline{n}$. So $y(t) \preceq x(t)$ for all $t \in I_x \cap I_y$. If $T_x < T_y$ instead, proceeding in the same manner as in the previous case we obtain $y(t) \preceq x(t)$ for all $t \in [t_0, T_x)$. If $I_x \cap I_y = [t_0, T_x]$, then $y(T_x) \preceq x(T_x)$ follows by continuity and taking the limit as $t \to T_x^-$. ■

C. Componentwise bounding procedure

The main result of this paper is the following, which establishes a method for bounding the magnitude vector ρ in a componentwise manner.

Proposition 3.1: Let z denote a solution to (1) with an arbitrary disturbance satisfying (2), and define $\rho = |z|$ (componentwise magnitude). Consider the real dynamic system (6). Let $[0, T_x)$ denote the maximal (forward) interval of existence of x. Then, $\rho(t) \preceq x(t)$ for all $t \in [0, T_x)$.

By means of Proposition 3.1, the obtention of componentwise magnitude bounds is reduced to the computation of a single and well-specificed trajetory of the bounding system (6). In this regard, the asymptotic behaviour of (1) could be estimated by analyzing that of (6). The idea of estimating the behaviour of all possible trajectories by means of a single trajectory from a comparison system has been previously employed in [16] in the context of stability of large-scale interconnections of nonlinear systems.

For the proof of Proposition 3.1 we require the following lemma, which shows that the upper-right Dini derivative of the magnitude vector ρ satisfies a componentwise inequality involving the vector field $\mathcal{M}[f]$.

Lemma 3.4: Let z denote a solution to (1), where the disturbance vector w satisfies (2). Let $[0, T)$ be the maximal interval of existence of z, and let $\rho = |z|$. Then,

$$D^+\rho(t) \preceq \mathcal{M}[f](\rho(t)) + \mathbf{w}, \quad \text{for all } t \in [0, T).$$

Proof: We have $\rho_i = z_i e^{-j\theta_i}$, where $\theta_i := \arg\{z_i\}$ is well-defined whenever $z_i \neq 0$, for all $i \in \underline{n}$. In addition, whenever $z_i \neq 0$, the argument θ_i can be selected so that

$$\dot{\rho}_i = \dot{z}_i e^{-j\theta_i} - j z_i e^{-j\theta_i}\dot{\theta}_i = \dot{z}_i e^{-j\theta_i} - j\rho_i\dot{\theta}_i.$$

Taking into account that $\rho_i, \theta_i \in \mathbb{R}$, then

$$\dot{\rho}_i = \mathbb{R}\mathrm{e}\{\dot{z}_i e^{-j\theta_i} - j\rho_i\dot{\theta}_i\} = \mathbb{R}\mathrm{e}\{\dot{z}_i e^{-j\theta_i}\}, \quad \text{if } z_i \neq 0. \tag{11}$$

Due to the continuous differentiability of the vector field f, and the fact that $f(0) = 0$, each of its components f_i can be represented as follows:

$$f_i(z) = \int_0^1 \sum_{k=1}^n \left(\frac{\partial f_i}{\partial z_k}(\sigma z) z_k \right) d\sigma. \tag{12}$$

Using (1) and (11), it follows that whenever $z_i \neq 0$, then

$$\dot{\rho}_i = \mathbb{R}\mathrm{e}\{(f_i(z) + \mathbf{w}_i)e^{-j\theta_i}\} \leq \mathbb{R}\mathrm{e}\{f_i(z)e^{-j\theta_i}\} + \mathbf{w}_i.$$

From (12), we have

$$\begin{aligned} \mathbb{R}\mathrm{e}\{f_i(z)e^{-j\theta_i}\} &= \mathbb{R}\mathrm{e}\left\{ \int_0^1 \sum_{k=1}^n \left(\frac{\partial f_i}{\partial z_k}(\sigma z) z_k \right) e^{-j\theta_i} d\sigma \right\} \\ &\leq \int_0^1 \mathbb{R}\mathrm{e}\left\{ \frac{\partial f_i}{\partial z_i}(\sigma z) \right\} \rho_i + \sum_{k=1, k\neq i}^n \left| \frac{\partial f_i}{\partial z_k}(\sigma z) \right| \rho_k d\sigma \\ &\leq \int_0^1 \max_{y \in \mathcal{V}_i(\sigma\rho)} \left[\mathcal{M}_i\left(\frac{\partial f}{\partial z}(y) \right) \rho \right] d\sigma = \mathcal{M}[f]_i(\rho), \end{aligned}$$

where the last inequality follows because $\sigma z \in \mathcal{V}_i(\sigma\rho)$ for all $\sigma \geq 0$. If $z_i(t) = 0$, then $\rho_i(t) = 0$ and

$$\begin{aligned} D^+\rho_i(t) &= \limsup_{h\to 0^+} \frac{\rho_i(t+h) - \rho_i(t)}{h} = \limsup_{h\to 0^+} \frac{\rho_i(t+h)}{h} \\ &= \limsup_{h\to 0^+} \left| \frac{z_i(t+h)}{h} \right| = |\dot{z}_i(t)| \leq |f_i(z(t))| + \mathbf{w}_i. \end{aligned}$$

Also, using (12), if $z_i(t) = 0$, at time t we have

$$\begin{aligned} |f_i(z)| &= \left| \int_0^1 \sum_{k=1}^n \left(\frac{\partial f_i}{\partial z_k}(\sigma z) z_k \right) d\sigma \right| \\ &\leq \int_0^1 \sum_{k=1, k\neq i}^n \left| \frac{\partial f_i}{\partial z_k}(\sigma z) \right| \rho_k \, d\sigma \leq \mathcal{M}[f]_i(\rho). \end{aligned}$$

Combining the obtained inequalities and the fact that whenever $\dot{\rho}_i$ exists, then $\dot{\rho}_i = D^+\rho_i$, the result follows. ■

Proof of Proposition 3.1: Let T_z be the maximal interval of existence of z. By Lemma 3.4, we know that $D^+\rho(t) \preceq \mathcal{M}[f](\rho(t)) + \mathbf{w}$ for all $t \in [0, T_z)$. By Lemma 3.2, we know that $\mathcal{M}[f]$ is of type K, and hence so is $g = \mathcal{M}[f] + \mathbf{w}$.

Claim 3: $x(t) \in \mathbb{R}^n_{\geq 0}$ for all $t \in [0, T_x)$.

Proof of Claim 3: For every $m \in \mathbb{N}$, consider the following auxiliary initial value problem

$$\dot{x}^m = \mathcal{M}[f](x^m) + \mathbf{w} + \frac{1}{m}\mathbf{1}, \quad x^m(0) = x(0) + \frac{1}{m}\mathbf{1},$$

defined in all of $\mathbb{R}^n$. Let $[0, T_m)$ be the maximal interval of existence of x^m. The trajectory x^m begins in the positive orthant because $x(0) \succeq 0$. We next show that $x^m(t) \in \mathbb{R}^n_{\geq 0}$ for all $t \in [t_0, T_m)$. By continuity of x^m, if x^m leaves $\mathbb{R}^n_{\geq 0}$ before ceasing to exist, there must exist $k \in \underline{n}$ and $s \in [0, T_m)$ such that $x_k^m(s) = 0$ and $x^m(s) \succeq 0$. Then, we would have

$$\dot{x}_k^m(s) = \mathcal{M}[f]_k(x^m(s)) + \mathbf{w}_k + \frac{1}{m} \geq \frac{1}{m} > 0,$$

where we have used the facts that $x^m(s) \in \mathbb{R}^n_{\geq 0}$, $\mathcal{M}[f]$ is of type K in $\mathbb{R}^n_{\geq 0}$, and $\mathcal{M}[f](0) = 0$. It is clear that, having a positive derivative, x_k^m cannot become negative. We have thus established that $x^m(t) \in \mathbb{R}^n_{\geq 0}$ for all $t \in [t_0, T_m)$. Then, by using standard results about the continuity of solutions of ODEs with respect to parameters and initial conditions (e.g. [10, Theorem 3.5]), it follows that for all $t \in T_x$, $x_m(t) \to x(t)$ as $m \to \infty$ and, in consequence, that $x(t) \in \mathbb{R}^n_{\geq 0}$. ◦

Applying Lemma 3.3 with $D = \mathbb{R}^n_{\geq 0}$, then $\rho(t) \preceq x(t)$ for all $t \in [0, T)$, with $T = \min\{T_x, T_z\}$. Since $\rho(t) = |z(t)| \succeq 0$, it follows that $T_z \geq T_x$ and hence $T = T_x$. Otherwise, $z(t)$ would lie in a compact set for all $t \in [0, T_z)$, hence $z(t)$ could be extended to $[0, T_z + \delta)$ for some $\delta > 0$, contradicting the fact that $[0, T_z)$ is its maximal interval of existence. ■

IV. Triangular systems: stability

In this section, we show that if the system is in triangular form and the origin is a locally exponentially stable equilibrium, then the same holds for the bounding system. Consider the following triangular system

$$\begin{array}{rcl} \dot{z}_1 & = & f_1(z_1, z_2, \dots, z_n), \\ \dot{z}_2 & = & f_2(z_2, \dots, z_n), \\ \vdots & \vdots & \vdots \\ \dot{z}_n & = & f_n(z_n). \end{array} \tag{13}$$

We assume that the linearization of system (13) about 0 is locally exponentially stable.

Assumption 1: $\mathbb{R}\mathrm{e}\left\{\frac{\partial f_i}{\partial z_i}(0)\right\} < 0$ for all $i \in \underline{n}$.

Proposition 4.1: Consider the triangular system (13) and let Assumption 1 hold. Then there exist positive constants δ, c and β such that every solution x to

$$\dot{x} = \mathcal{M}[f](x), \quad \text{with } 0 \preceq x(0) \preceq \delta \mathbf{1}, \tag{14}$$

satisfies $x(t) \succeq 0$ for all $t \geq 0$ and

$$\|x(t)\| \leq c\|x(0)\|e^{-\beta t}, \quad \forall t \geq 0. \tag{15}$$

Proof: Let $A = \mathcal{M}\left(\frac{\partial f}{\partial z}(0)\right)$, define

$$e(x) := \mathcal{M}[f](x) - A\,|x|, \tag{16}$$

and let A_i denote the i-th row of A. We next show that $\lim_{x\to 0} \frac{e_i(x)}{\|x\|} = 0$ for every $i \in \underline{n}$. We have

$$\begin{aligned} e_i(x) &= \mathcal{M}[f]_i(x) - A_i|x| \\ &= \int_0^1 \left\{ \max_{y\in\mathcal{V}_i(\sigma|x|)} \left[\mathcal{M}_i\left(\frac{\partial f}{\partial z}(y)\right)|x|\right] - A_i|x| \right\} d\sigma \\ &= \int_0^1 \max_{y\in\mathcal{V}_i(\sigma|x|)} \left\{ \left[\mathcal{M}_i\left(\frac{\partial f}{\partial z}(y)\right) - A_i\right]|x| \right\} d\sigma \\ &= \int_0^1 \max_{y\in\mathcal{V}_i(\sigma|x|)} \left[\sum_{k=1}^n b_{i,k}(y)|x_k|\right] d\sigma \end{aligned} \tag{17}$$

with $b_{i,k}(y) = |\partial f_i/\partial z_k(y)| - |\partial f_i/\partial z_k(0)|$ if $i \neq k$ and $b_{i,i}(y) = \mathbb{R}\mathrm{e}\{\partial f_i/\partial z_i(y) - \partial f_i/\partial z_i(0)\}$. From the continuity of the partial derivatives of f and the fact that $\mathcal{V}_i(\sigma|x|) \subset \{y \in \mathbb{F}^n : \|y\| \leq \|x\|\} =: B(\|x\|)$ for all $0 \leq \sigma \leq 1$, it easily follows that $\max_{y\in\mathcal{V}_i(\sigma|x|)} |b_{i,k}(y)| \leq \max_{y\in B(\|x\|)} |b_{i,k}(y)| =: \gamma_{i,k}(x)$ and that $\gamma_{i,k}(x) \to 0$ as $x \to 0$. Then, using (17), we have

$$|e_i(x)| \leq \sum_{k=1}^n \gamma_{i,k}(x)|x_k| \quad \text{and hence} \quad \lim_{x\to 0} \frac{|e_i(x)|}{\|x\|} = 0$$

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be defined via

$$g(x) = Ax + e(x). \tag{18}$$

By the above derivation, it follows that g is differentiable at the origin and $\frac{\partial g}{\partial x}(0) = A$. Note that g coincides with $\mathcal{M}[f](x)$ whenever $x \in \mathbb{R}^n_{\geq 0}$, and that A is Hurwitz because (a) system (13) is triangular and hence A is upper triangular, and (b) by Assumption 1 the diagonal entries of A have negative real part. It thus follows that $\dot{x} = g(x)$ has a locally exponentially stable equilibrium at the origin.

The proof concludes by taking into account that the solutions of (14) that begin in $\mathbb{R}^n_{\geq 0}$ remain in $\mathbb{R}^n_{\geq 0}$ while they exist (see the proof of Proposition 3.1) and that, in consequence, are also solutions to $\dot{x} = g(x)$. ■

V. Examples

We provide two examples to illustrate the procedure and some of its features. Both examples have the same equations, the first interpreted as a complex system and the second as a real system. The equations correspond to the second-order triangular system (1)–(2), with $|z(0)| = [1, 2]'$, and

$$\begin{aligned} f_1(z_1, z_2) &:= -(1 + z_2^2)z_1, & |w_1(t)| &\leq \mathbf{w}_1 := 0, \\ f_2(z_1, z_2) &:= -z_2, & |w_2(t)| &\leq \mathbf{w}_2 := 1. \end{aligned}$$

We can compute the Jacobian

$$\frac{\partial f}{\partial z}(y) = \begin{bmatrix} -(1 + y_2^2) & -2y_1y_2 \\ 0 & -1 \end{bmatrix}.$$

A. Complex system

We allow $z(t) \in \mathbb{C}^2$. From (5), for $\sigma \in [0, 1]$ we have

$$\mathcal{V}_1(\sigma|x|) = \{y \in \mathbb{C}^2 : |y_1| = \sigma|x_1|, |y_2| \leq \sigma|x_2|\}$$

and hence

$$\begin{aligned} &\max_{y\in\mathcal{V}_1(\sigma|x|)} \left[\mathcal{M}_1\left(\frac{\partial f}{\partial z}(y)\right)|x|\right] \\ &= \max_{y\in\mathcal{V}_1(\sigma|x|)} \left[\mathbb{R}\mathrm{e}\{-(1 + y_2^2)\}|x_1| + |-2y_1y_2||x_2|\right] \\ &= \max_{|y_2|\leq\sigma|x_2|} \left[-(1 + \mathbb{R}\mathrm{e}\{y_2^2\})|x_1| + 2\sigma|x_1||y_2||x_2|\right] \\ &= \max_{\substack{|y_2|\leq\sigma|x_2| \\ \theta_2\in[0,2\pi]}} \left[(-1 - |y_2|^2\cos(2\theta_2))|x_1| + 2\sigma|x_1||y_2||x_2|\right] \\ &= \max_{|y_2|\leq\sigma|x_2|} \left[(-1 + |y_2|^2)|x_1| + 2\sigma|x_1||y_2||x_2|\right] \\ &= (-1 + \sigma^2|x_2|^2)|x_1| + 2\sigma^2|x_1||x_2|^2 \\ &= (-1 + 3\sigma^2|x_2|^2)|x_1|. \end{aligned}$$

From (4), integrating the above expression we get

$$\mathcal{M}[f]_1(x) = (-1 + |x_2|^2)|x_1|.$$

Similarly, it follows that

$$\mathcal{M}[f]_2(x) = -|x_2|. \tag{19}$$

The initial condition $z_0 = [1,\ 2j]'$ (with $j^2 = -1$) and the constant disturbance $w_1(t) = 0$, $w_2(t) = j$ for all $t \geq 0$ (which satisfies $|w(t)| = \mathbf{w}$), produce

$$\begin{aligned} &z_2(t) = (1 + e^{-t})j, \\ &\dot{z}_1(t) = -(1 - (1 + e^{-t})^2)z_1(t) = (2e^{-t} + e^{-2t})z_1(t), \end{aligned}$$

$$z_1(t) = e^{\int_0^t (2e^{-s}+e^{-2s})ds} z_1(0)$$

and taking $x(0) = |z(0)| = [1,\ 2]'$ and $\mathbf{w} = [0,\ 1]'$, then the solution to (6) is given by

$$\begin{aligned} x_2(t) &= 1 + e^{-t} = |z_2(t)|, \\ \dot{x}_1(t) &= (-1 + (1 + e^{-t})^2) x_1(t) = (2e^{-t} + e^{-2t}) x_1(t), \\ x_1(t) &= e^{\int_0^t (2e^{-s}+e^{-2s})ds} x_1(0) \end{aligned}$$

and it is clear that $x_1(t) = |z_1(t)|$. Hence, in this example there is no conservativeness in the bounding procedure.

B. Real system

Next, we interpret $z(t) \in \mathbb{R}^2$ and hence $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $\mathcal{V}_1(\sigma|x|) = \{y \in \mathbb{R}^2 : |y_1| = \sigma|x_1|, |y_2| \le \sigma|x_2|\}$. Then,

$$\begin{aligned} &\max_{y \in \mathcal{V}_1(\sigma|x|)} \left[\mathcal{M}_1 \left(\frac{\partial f}{\partial z}(y) \right) |x| \right] \\ &\quad= \max_{y \in \mathcal{V}_1(\sigma|x|)} \left[\mathbb{R}\mathrm{e}\{-(1 + y_2^2)\}|x_1| + |-2y_1 y_2||x_2| \right] \\ &\quad= \max_{|y_2| \le \sigma|x_2|} \left[-(1 + |y_2|^2)|x_1| + 2\sigma|x_1||y_2||x_2| \right] \\ &\quad= -|x_1| + \max_{|y_2| \le \sigma|x_2|} \left[-|y_2|^2 + 2\sigma|y_2||x_2| \right] |x_1|. \end{aligned}$$

The maximum of the latter expression between square brackets is attained for $|y_2| = \sigma|x_2|$. Therefore,

$$\max_{y \in \mathcal{V}_1(\sigma|x|)} \left[\mathcal{M}_1 \left(\frac{\partial f}{\partial z}(y) \right) |x| \right] = -|x_1| + \sigma^2 |x_2|^2 |x_1|.$$

Integrating the above expression we get

$$\mathcal{M}[f]_1(x) = \left(-1 + \frac{1}{3}|x_2|^2 \right) |x_1|,$$

and (19) follows similarly. For the initial condition $z(0) = [1, 2]' = |z(0)| = x(0)$, we can compute $z_2(t) = 1 + e^{-t} = x_2(t)$ and $\dot{z}_1(t) = -(1 + (1 + e^{-t})^2) z_1(t)$, whence

$$\begin{aligned} z_1(t) &= e^{\int_0^t -(1+(1+e^{-s})^2)ds} z_1(0), \\ x_1(t) &= e^{\int_0^t -(1+(1+e^{-s})^2/3)ds} x_1(0). \end{aligned}$$

By contrast with the complex case, the transient bound on $|z_1|$, namely x_1, exhibits some conservativeness although in this specific case the asymptotic bound is tight, i.e. $\limsup_{t\to\infty} |z_1(t)| = \limsup_{t\to\infty} x_1(t)$.

VI. Conclusions

We have developed a novel componentwise magnitude bounding procedure applicable to nonlinear systems with additive disturbances. Every system trajectory beginning at a given initial condition is shown to be bounded componentwise by a single trajectory from a bounding system, irrespective of the disturbance evolution. This bounding system is monotone and positive, and is constructed based on the original system's equations and disturbance bounds. For LTI systems, we have shown that this bounding procedure reduces to a previously existing one. We have also provided preliminary results showing that the bounding system is able to maintain local exponential stability of the origin if the system is triangular. Future work will be aimed at extending the procedure to more general classes of systems, and to the analysis of conditions that ensure other types of stability properties, such as global asymptotic stability and input-to-state stability (ISS) from the disturbance.

Acknowledgment

Work partially supported by ANPCyT grants PICT 2013-0852 and PICT 2014-2599, Argentina.

References

[1] D. Angeli and E. D. Sontag. Monotone control systems. *IEEE Trans. on Automatic Control*, 48(10):1684–1698, 2003.

[2] G. Dirr, H. Ito, A. Rantzer, and B. S. Rüffer. Separable Lyapunov functions for monotone systems: constructions and limitations. *Discrete and Continuous Dynamical Systems - Series B*, 20(8):2497–2526, 2015.

[3] R. A. Freeman and P. V. Kokotović. *Robust nonlinear control design: state-space and Lyapunov techniques*. Birkhäuser Verlag AG, 1996.

[4] H. Haimovich, E. Kofman, and M. M. Seron. Systematic ultimate bound computation for sampled-data systems with quantization. *Automatica*, 43(6):1117–1123, 2007.

[5] H. Haimovich and M.M. Seron. Componentwise ultimate bound computation for switched linear systems. In *Proc. 48th IEEE Conference on Decision and Control, Shanghai, China*, pages 2150–2155, 2009.

[6] H. Haimovich and M.M. Seron. Componentwise ultimate bound and invariant set computation for switched linear systems. *Automatica*, 46(11):1897–1901, 2010.

[7] H. Haimovich and M.M. Seron. Bounds and invariant sets for a class of switching systems with delayed-state-dependent perturbations. *Automatica*, 49(3):748–754, 2013.

[8] H. Haimovich and M.M. Seron. Bounds and invariant sets for a class of discrete-time switching systems with perturbations. *International Journal of Control*, 87(2):371–383, 2014.

[9] S. Huang, M.R. James, D. Nešić, and P.M. Dower. Analysis of input to state stability for discrete-time nonlinear systems via dynamic programming. *Automatica*, 41(12):2055–2065, 2005.

[10] H. Khalil. *Nonlinear Systems*. Prentice-Hall, New Jersey, 3rd edition, 2002.

[11] E. Kofman, J. A. De Dona, and M. M. Seron. Probabilistic set invariance and ultimate boundedness. *Automatica*, 48(10):2670–2676, 2012.

[12] E. Kofman, J. A. De Dona, M. M. Seron, and N. Pizzi. Continuous-time probabilistic ultimate bounds and invariant sets: computation and assignment. *Automatica*, 71:98–105, 2016.

[13] E. Kofman, H. Haimovich, and M. M. Seron. A systematic method to obtain ultimate bounds for perturbed systems. *International Journal of Control*, 80(2):167–178, 2007.

[14] H. Li, R. Baier, L. Grüne, S. F. Hafstein, and F. R. Wirth. Computation of local ISS lyapunov functions with low gains via linear programming. *Discrete and Continuous Dynamical Systems - Series B*, 20(8):2477–2495, 2015.

[15] S. G. Nersesov and W. M. Haddad. On the stability and control of nonlinear dynamical systems via vector Lyapunov functions. *IEEE Trans. on Automatic Control*, 51(2):203–215, 2006.

[16] B. Rüffer, H. Ito, and P. M. Dower. Computing asymptotic gains of large-scale interconnections. In *Proc. 49th IEEE Conf. on Decision and Control, Atlanta, GA, USA*, pages 7413–7418, 2010.

[17] Hal L. Smith. *Monotone dynamical systems: an introduction to the theory of competitive and cooperative systems*. American Mathematical Society, Providence, Rhode Island, 1995.

[18] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. on Automatic Control*, 34:435–443, 1989.

[19] E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. *Systems and Control Letters*, 24:351–359, 1995.

[20] H. Zhang and P. M. Dower. Minimum integral input-to-state stability bounds. In *1st Australian Control Conference, Melbourne, Australia*, pages 76–81, 2011.

[21] H. Zhang and P. M. Dower. Performance bounds for nonlinear systems with a nonlinear $\mathcal{L}_2$-gain property. *International Journal of Control*, 85(9):1293–1312, 2012.

[22] H. Zhang and P. M. Dower. Computation of tight integral input-to-state stability bounds for nonlinear systems. *Systems and Control Letters*, 62:355–365, 2013.