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Some Invariance Principles for constrained Switched Systems ⋆

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Abstract: In this paper we consider switched nonlinear systems under average dwell time switching signals, with an otherwise arbitrary compact index set and with additional constraints in the switchings. We present invariance principles for these systems and derive by using observability-like notions some convergence and asymptotic stability criteria. These results may enable us to analyze the stability of solutions of switched systems with both state-dependent constrained switching and switching whose logic has memory, i.e., the active subsystem only can switch to a prescribed subset of subsystems.

Keywords: switched systems; observability; stability; convergence; invariance.

1. INTRODUCTION

In the last few years, various extensions to switched systems of LaSalle's invariance principle for differential equations (see [8, 9]) were obtained, that enable us to determine the convergence of the solutions of a switching system to an equilibrium point, and consequently the asymptotic stability. Hespanha in [6] introduced an invariance principle for switched linear systems under *persistently dwell-time* switching signals and in [7] Hespanha *et al.* extended some of those results to a family of nonlinear systems. Bacciotti and Mazzi presented in [1] an invariance principle for switched systems with *dwell-time* signals. An invariance principle for switched nonlinear systems with *average dwell-time* signals that satisfy *state-dependent* constraints was derived by Mancilla-Aguilar and García in [14] from the sequential compactness of particular classes of trajectories of switched systems. Based on invariance results for hybrid systems ([16]), Goebel *et al.* in [5] obtained recently invariance results for switched systems under various types of switching signals. Lee and Jiang in [10] gave a generalized version of Krasovskii-LaSalle Theorem for time-varying switched systems. Under certain ergodicity conditions on the switching signal, some stability results were also obtained in [3, 17, 18].

Most of the invariance results for switched systems already published only consider restrictions originated by the timing of the switchings or by the state dependence of it. Nevertheless there is also an important restriction to take into account: the fact that not all the subsystems may be accessible from a particular one, i.e. the case in which the switching logic has memory. This restriction is clearly exhibited, for example, in switched systems which are the continuous portion of a hybrid automaton (see [4], [13]). In this regard, the invariance principles developed for hybrid systems in [13] and in [16] could be useful in the analysis of switched systems with this class of restriction in the switchings.

In this paper we present invariance results that hold for trajectories of switched systems with a non necessarily finite number of subsystems and whose switching signals verify an average dwell time condition and belong to a family for which a certain property **P** holds. As various of the restricted switching classes mentioned above satisfy **P**, these results enable us to obtain in an unified way invariance theorems for all of them. Based on these invariance results, we derive new convergence and stability criteria that recover, generalize and strengthen some results previously obtained. Due to the length restriction, we have omitted most of the proofs. For more detailed discussions, proofs and additional results see [15].

The article unfolds as follows. Section 2. contains the basic definitions. In Section 3. we present invariance principles for switched systems with constrained switching. Convergence and stability results are given in Section 4. Finally Section 5. contains some conclusions.

2. BASIC DEFINITIONS

In this work we consider switched systems described by

$$\dot{x} = f(x, \sigma) \tag{1}$$

where x takes values in $\mathbb{R}^n$, $\sigma : \mathbb{R} \to \Gamma$, with Γ a compact metric space, is a *switching signal*, *i.e.*, σ is piecewise constant (it has at most a finite number of jumps in each compact interval) and is continuous from the right and $f : \mathrm{dom}(f) \to \mathbb{R}^n$, with $\mathrm{dom}(f)$ a closed subset of $\mathbb{R}^n \times \Gamma$, is continuous.

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For each $\gamma \in \Gamma$, let $\chi_\gamma = \{\xi \in \mathbb{R}^n : (\xi, \gamma) \in \mathrm{dom}(f)\}$ and $f_\gamma : \chi_\gamma \to \mathbb{R}^n$ be defined by $f_\gamma(\xi) = f(\xi, \gamma)$; then χ_γ is closed and f_γ is continuous for any $\gamma \in \Gamma$. We note that when Γ is finite, these last two conditions imply that $\mathrm{dom}(f)$ is closed and that f is continuous on $\mathrm{dom}(f)$. In the sequel we denote with $\mathcal{S}$ the set of all the switching signals.

Given $\sigma \in \mathcal{S}$, a solution of (1) corresponding to σ is a locally absolutely continuous function $x : I_x \to \mathbb{R}^n$, with $I_x \subset \mathbb{R}$ a nonempty interval, such that $(x(t), \sigma(t)) \in \mathrm{dom}(f)$ for all $t \in I_x$ and $\dot{x}(t) = f(x(t), \sigma(t))$ for almost all $t \in I_x$. The solution x is complete if $I_x = \mathbb{R}$ and forward complete if $\mathbb{R}_{\geq 0} \subset I_x$. A pair (x, σ) is a trajectory of (1) if $\sigma \in \mathcal{S}$ and x is a solution of (1) corresponding to σ. The trajectory is complete or forward complete if x is complete or forward complete, respectively. Given a subset $\mathcal{O}$ of $\mathbb{R}^n$, we say that the trajectory (x, σ) is precompact relative to $\mathcal{O}$ if there exists a compact set $B \subset \mathcal{O}$ such that $x(t) \in B$ for all $t \in I_x$. When $\mathcal{O} = \mathbb{R}^n$ we simply say that (x, σ) is precompact.

Remark 2.1. Note that we do not suppose that $\mathrm{dom}(f) = \mathbb{R}^n \times \Gamma$. In this way we can take into account, in the analysis of the asymptotic behavior of a given trajectory (x, σ) of (1), some kind of state-dependent constraints which the trajectory under study must satisfy (see [15]). By doing so we can consider the system as if its switching is state-independent, and focus on the restrictions imposed to it by the timing of the discontinuities of the switching signal and/or by the accessibility to certain subsystems from another ones.

In this paper we consider forward complete solutions of (1) corresponding to switching signals σ which belong to particular subclasses of $\mathcal{S}$. Let $\Lambda(\sigma)$ be the set of times where σ has a jump (switching times). Following [6] we say that $\sigma \in \mathcal{S}$ has a dwell-time $\tau_D > 0$ if $|t - t'| \geq \tau_D$ for any pair $t, t' \in \Lambda(\sigma)$ such that $t \neq t'$.

A switching signal σ has an average dwell-time $\tau_D > 0$ and a chatter bound $N_0 \in \mathbb{N}$ if the number of switching times of σ in any open finite interval $(\tau_1, \tau_2) \subset \mathbb{R}$ is bounded by $N_0 + (\tau_2 - \tau_1)/\tau_D$, i.e. $\mathrm{card}(\Lambda(\sigma) \cap (\tau_1, \tau_2)) \leq N_0 + (\tau_2 - \tau_1)/\tau_D$.

We denote by $\mathcal{S}_a[\tau_D, N_0]$ the set of all the switching signals which have an average dwell-time $\tau_D > 0$ and a chatter bound $N_0 \in \mathbb{N}$ and by $\mathcal{T}_a[\tau_D, N_0]$ the set of all the complete trajectories (x, σ) of (1) with $\sigma \in \mathcal{S}_a[\tau_D, N_0]$ and let $\mathcal{S}_a = \bigcup_{\tau_D > 0, N_0 > 0} \mathcal{S}_a[\tau_D, N_0]$ and $\mathcal{T}_a = \bigcup_{\tau_D > 0, N_0 > 0} \mathcal{T}_a[\tau_D, N_0]$. We note that the set of switching signals σ which have a dwell-time $\tau_D > 0$ coincides with $\mathcal{S}_a[\tau_D, 1] := \mathcal{S}_d[\tau_D]$. We denote by $\mathcal{T}_d[\tau_D]$ the set of all the complete trajectories (x, σ) of (1) with $\sigma \in \mathcal{S}_d[\tau_D]$ and let $\mathcal{S}_d = \bigcup_{\tau_D > 0} \mathcal{S}_d[\tau_D]$ and $\mathcal{T}_d = \bigcup_{\tau_D > 0} \mathcal{T}_d[\tau_D]$.

For Γ a finite set and $T > 0$, we denote by $\mathcal{S}_e[T]$ the family of all the switching signals σ which verify the following "ergodicity" condition: for every $t_0 \geq 0$ and every $\gamma \in \Gamma$, $\sigma^{-1}(\gamma) \cap [t_0, t_0 + T] \neq \emptyset$.

$\mathcal{T}_e[T]$ will denote the set of complete trajectories (x, σ) with $\sigma \in \mathcal{S}_e[T]$ and $\mathcal{S}_e = \bigcup_{T > 0} \mathcal{S}_e[T]$ and $\mathcal{T}_e = \bigcup_{T > 0} \mathcal{T}_e[T]$.

The families of switching signals already introduced have no restrictions on the accessibility from any subsystem to another. The family of switching signals —and their corresponding trajectories— that we introduce next, takes into account the case in which the switching logic has memory, i.e. when a subsystem corresponding to an index $\gamma \in \Gamma$ can only switch to subsystems corresponding to modes γ' that belong to a certain subset $\Gamma_\gamma \subset \Gamma$.

Given a set-valued map $H : \Gamma \rightsquigarrow \Gamma$, $\mathcal{S}^H$ is the set of all the switching signals σ which verify the condition $\sigma(t) \in H(\sigma(t^-))$ for every time $t \in \Lambda(\sigma)$. Here $\sigma(t^-) = \lim_{s \to t^-} \sigma(s)$. $\mathcal{T}^H$ denotes the set of all the complete trajectories (x, σ) with $\sigma \in \mathcal{S}^H$. This class of switching signals enable us, for example, to model the restrictions imposed by the discrete process of a hybrid system whose continuous portion is as in (1) (see [4]).

3. INVARIANCE RESULTS

In this section we present some invariance results that enable us to characterize the asymptotic behavior of a precompact forward complete trajectory (x, σ) of (1) with σ belonging to a certain subclass of $\mathcal{S}_a$. The consideration of such subclass allows us to obtain in an unified way invariance results for systems whose switching signals undergo different restrictions.

We recall that a point $\xi \in \mathbb{R}^n$ belongs to $\Omega(x)$, the ω-limit set of $x : I_x \to \mathbb{R}^n$, with $\mathbb{R}_{\geq 0} \subset I_x$, if there exists a strictly increasing sequence of times $\{s_k\} \subset I_x$ with $\lim_{k\to\infty} s_k = +\infty$ and $\lim_{k\to\infty} x(s_k) = \xi$. The ω-limit set $\Omega(x)$ is always closed and, when x evolves in a compact set of $\mathbb{R}^n$, it is nonempty, compact, connected if x is continuous, and $x \to \Omega(x)$ (for a set $M \subset \mathbb{R}^n$, $x \to M$ if $\lim_{t\to+\infty} d(x(t), M) = 0$, being $d(\xi, M) = \inf_{\nu \in M} |\nu - \xi|$).

As was done in [14], we will associate to each forward complete trajectory (x, σ) of (1) with $\sigma \in \mathcal{S}_a$, the nonempty set $\Omega^\sharp(x, \sigma) \subset \mathbb{R}^n \times \Gamma$ that we introduce in the following

Definition 3.1. Given a forward complete trajectory (x, σ) of (1) with $\sigma \in \mathcal{S}_a$, a point $(\xi, \gamma) \in \mathbb{R}^n \times \Gamma$ belongs to $\Omega^\sharp(x, \sigma)$ if there exists a strictly increasing and unbounded sequence $\{s_k\} \subset \mathbb{R}_{\geq 0}$ such that

(1) $\lim_{k\to\infty} \tau^1_\sigma(s_k) - s_k = r$, $0 < r \leq \infty$,
(2) $\lim_{k\to\infty} x(s_k) = \xi$ and $\lim_{k\to\infty} \sigma(s_k) = \gamma$.

Here, for any $t \in \mathbb{R}$, $\tau^1_\sigma(t) = \inf\{s \in \Lambda(\sigma) : t < s\}$ if $\{s \in \Lambda(\sigma) : t < s\} \neq \emptyset$ and $\tau^1_\sigma(t) = +\infty$ in other case (i.e. $\tau^1_\sigma(t)$ is the first switching time greater than t).

Let $\pi_1 : \mathbb{R}^n \times \Gamma \to \mathbb{R}^n$ be the projection onto the first component. Then the following relation between $\Omega(x)$ and $\Omega^\sharp(x, \sigma)$ holds.

Lemma 1. Let (x, σ) be a forward complete trajectory of (1) with $\sigma \in \mathcal{S}_a$ that is precompact relative to $\mathcal{O} \subset \mathbb{R}^n$. Then $\Omega^\sharp(x, \sigma) \subset \mathrm{dom}(f) \cap (\mathcal{O} \times \Gamma)$ and $\Omega(x) = \pi_1(\Omega^\sharp(x, \sigma))$.

In order to see that the set $\Omega^\sharp(x, \sigma)$ enjoys certain kind of invariance property, let us introduce the following

Definition 3.2. Given a family $\mathcal{T}^*$ of complete trajectories of (1), we say that a nonempty subset $M \subset \mathbb{R}^n \times \Gamma$ is *weakly-invariant w.r.t* $\mathcal{T}^*$ if for each $(\xi, \gamma) \in M$ there is a trajectory $(x,\sigma) \in \mathcal{T}^*$ such that $x(0) = \xi$, $\sigma(0) = \gamma$ and $(x(t), \sigma(t)) \in M$ for all $t \in \mathbb{R}$.

This notion of weak invariance differs from the one introduced in [14], in that the last one involves only forward invariance while the introduced here also involves backward invariance.

Let us introduce now the following class of switching signals.

Definition 3.3. We say that a family of switching signals $\mathcal{S}^*$ has the property $\mathbf{P}$ if

(1) $\mathcal{S}^* \subset \mathcal{S}_a[\tau_D, N_0]$ for some $\tau_D > 0$ and some $N_0 \in \mathbb{N}$;
(2) for any $s > 0$ and any $\sigma \in \mathcal{S}^*$, $\sigma(\cdot + s) \in \mathcal{S}^*$;
(3) for every sequence $\{\sigma_k\} \subset \mathcal{S}^*$, there exist $\sigma^* \in \mathcal{S}^*$ and a subsequence $\{\sigma_{k_l}\}$ such that $\lim_{l\to\infty} \sigma_{k_l}(t) = \sigma^*(t)$ for almost all $t \in \mathbb{R}$.

Lemma 2. The following classes of switching signals have the property $\mathbf{P}$:

(1) $\mathcal{S}_a[\tau_D, N_0]$ for every $\tau_D > 0$ and every $N_0 \in \mathbb{N}$;
(2) $\mathcal{S}_d[\tau_D] \cap \mathcal{S}^H$ for all $\tau_D > 0$ and every $H : \Gamma \rightsquigarrow \Gamma$ such that the set $\text{Graph}(H) = \{(\gamma, \gamma') \in \Gamma \times \Gamma : \gamma' \in H(\gamma)\}$ is closed;
(3) $\mathcal{S}_d[\tau_D] \cap \mathcal{S}_e[T]$ for every $\tau_D > 0$ and every $T > 0$.

The next result will be instrumental in what follows.

Theorem 3. Let $\mathcal{S}^*$ be a family of switching signals which verifies property $\mathbf{P}$ and let $\mathcal{T}^*$ be the set of all the complete trajectories $(\overline{x}, \overline{\sigma})$ of (1) with $\overline{\sigma} \in \mathcal{S}^*$. Then, if (x,σ) is a precompact forward complete trajectory of (1) such that $\sigma \in \mathcal{S}^*$, $\Omega^\sharp(x,\sigma)$ is weakly-invariant w.r.t $\mathcal{T}^*$.

Remark 3.1. Since the weak invariance of $\Omega^\sharp(x,\sigma)$ is a cornerstone of the results that we present below (Theorems 4 and 5), Theorem 3 enables us to obtain in a unified way invariance results not only for the different switching signals explicitly mentioned in Lemma 2 but also for any other that verifies property $\mathbf{P}$.

Remark 3.2. At first glance, it would seem more natural to consider for a given precompact forward complete trajectory (x,σ) of (1) its ω-limit set

$$\Omega(x,\sigma) = \{(\xi,\gamma) : \exists t_k \uparrow \infty, (x(t_k), \sigma(t_k)) \to (\xi,\gamma)\},$$

instead of $\Omega^\sharp(x,\sigma) \subset \Omega(x,\sigma)$. Nevertheless, there exist forward complete trajectories (x,σ) of (1) with $\sigma \in \mathcal{S}_a$ such that $\Omega(x,\sigma)$ is not weakly-invariant for any family of trajectories of that switched system.

Next, we present two invariance results that involve the existence of a function V which is nonincreasing along a trajectory of (1). In order to do so, we introduce the following class of functions.

Definition 3.4. We say that a function $V : \text{dom}(V) \to \mathbb{R}$ belongs to class $\mathcal{V}$, if it verifies

(1) $\text{dom}(V) \subset \mathbb{R}^n \times \Gamma$.
(2) For every $\gamma \in \Gamma$, $\mathcal{D}_\gamma := \{\xi \in \mathbb{R}^n : (\xi,\gamma) \in \text{dom}(V)\}$ is an open set.
(3) Let $\mathcal{O} := \pi_1(\text{dom}(V))$. Then $\mathcal{O}_\gamma := \mathcal{O} \cap \chi_\gamma \subset \mathcal{D}_\gamma \ \forall \gamma \in \Gamma$
(4) For all $\gamma \in \Gamma$, $V_\gamma(\cdot) := V(\cdot, \gamma)$ is differentiable on $\mathcal{O}_\gamma$.

We note that $\text{dom}(f) \cap (\mathcal{O} \times \Gamma) = \cup_{\gamma\in\Gamma}(\mathcal{O}_\gamma \times \{\gamma\}) \subset \text{dom}(f) \cap \text{dom}(V)$.

We also note that when Γ is finite, the restriction of any function $V \in \mathcal{V}$ to $\text{dom}(f) \cap (\mathcal{O} \times \Gamma)$ is continuous.

In what follows, for a function $V \in \mathcal{V}$, let $Z_V = \{(\xi,\gamma) \in \text{dom}(f) \cap (\mathcal{O} \times \Gamma) : \nabla V_\gamma(\xi) f_\gamma(\xi) = 0\}$.

Assumption 1. The forward complete trajectory (x,σ) of (1) verifies the following: there exists a function $V \in \mathcal{V}$ whose restriction to $\text{dom}(f) \cap (\mathcal{O} \times \Gamma)$ is continuous, (x,σ) is precompact relative to $\mathcal{O}$ and $v(t) = V(x(t), \sigma(t))$ is nonincreasing on $[0, +\infty)$.

Theorem 4. Let $\mathcal{S}^*$ be a family of switching signals which has property $\mathbf{P}$ and let $\mathcal{T}^*$ be the set of all the complete trajectories (x,σ) of (1) with $\sigma \in \mathcal{S}^*$. Suppose that (x,σ), with $\sigma \in \mathcal{S}^*$, is a forward complete trajectory of (1) for which Assumption 1 holds. Then there exists $c \in \mathbb{R}$ such that $x \to \pi_1(M(c))$, where $M(c)$ is the maximal weakly-invariant set w.r.t. $\mathcal{T}^*$ contained in $V^{-1}(c) \cap Z_V$.

Remark 3.3. We note that Theorem 4 is an extension to switched systems of the well known LaSalle's invariance principle for differential equations (see, for example, [9, Theorem 6.4]).

In the sequel, for any $\sigma \in \mathcal{S}$ and any $\gamma \in \Gamma$, let $\mathcal{I}_{\sigma,\gamma} = \sigma^{-1}(\gamma) \cap [0, +\infty)$.

When Γ is a finite set, we can relax the nonincreasing condition in Assumption 1 as follows.

Assumption 2. The forward complete trajectory (x,σ) of (1) verifies the following: there exists a function $V \in \mathcal{V}$ such that (x,σ) is precompact relative to $\mathcal{O}$ and $v(t) = V(x(t), \sigma(t))$ is nonincreasing on $\mathcal{I}_{\sigma,\gamma}$, for all $\gamma \in \Gamma$.

Remark 3.4. Assumptions of this kind are standard when the stability analysis of the zero solution of a switched system is performed by means of multiple Lyapunov functions (see [4], [11]).

In what follows, when Γ is a finite set, we identify it with the set $\{1, \ldots, N\} \subset \mathbb{N}$, where $N = \text{card}(\Gamma)$.

Theorem 5. Suppose that Γ is finite and let $\mathcal{S}^*$ and $\mathcal{T}^*$ be as in Theorem 4. Suppose that (x,σ), with $\sigma \in \mathcal{S}^*$, is a forward complete trajectory of (1) for which Assumption 2 holds. Then there exists $\boldsymbol{c} = (c_1, \ldots, c_N) \in \mathbb{R}^N$ such that $x \to \pi_1(M(\boldsymbol{c}))$, where $M(\boldsymbol{c})$ is the maximal weakly-invariant set w.r.t. $\mathcal{T}^*$ contained in $\cup_{\gamma\in\Gamma}\{(\xi,\gamma) \in \text{dom}(f) \cap (\mathcal{O} \times \Gamma) : V_\gamma(\xi) = c_\gamma\} \cap Z_V$.

Remark 3.5. Some invariance results for switched systems reported in the literature can be derived from Theorem 5. In particular [1, Theorems 1 and 2], [14, Proposition 4.1] and [5, Corollary 5.6].

Remark 3.6. The proofs of theorems 4 and 5 follow from a) Theorem 3, b) the fact that $\Omega^\sharp(x,\sigma)$ is contained in $V^{-1}(c) \cap Z_V$ for some $c \in \mathbb{R}$ under the hypotheses of Theorem 4 and in $\cup_{\gamma\in\Gamma}\{(\xi,\gamma) \in \text{dom}(f) \cap (\mathcal{O} \times \Gamma) : V_\gamma(\xi) = c_\gamma\} \cap Z_V$ for some $\boldsymbol{c} \in \mathbb{R}^N$ when the hypotheses of Theorem 5 hold, and c) Lemma 1.

4. CONVERGENCE AND STABILITY RESULTS

In this section we give some convergence and stability results for switched systems with constrained switchings, whose proofs rely on Lemma 2 and the invariance principles presented in Section 3.

4.1 Convergence results

Let us first introduce some observability-like definitions.

Given a subset $\mathcal{X} \subset \mathbb{R}^n$, a continuous map $g : \mathcal{X} \to \mathbb{R}^n$ and a function $h : \mathcal{X} \to \mathbb{R}$, we say that for a given τ ($\tau > 0$ or $\tau = \infty$) a point $\xi \in \mathcal{X}$ belongs to the set $\mathcal{X}^f(g, h, \tau)$ (resp. $\mathcal{X}^b(g, h, \tau)$) if there exists a solution $\varphi : [0, \tau] \to \mathcal{X}$ (resp. $\varphi : [-\tau, 0] \to \mathcal{X}$) of $\dot{x} = g(x)$ such that $\varphi(0) = \xi$ and $h(\varphi(t)) = 0$ for all $t \in [0, \tau]$ (resp. $t \in [-\tau, 0]$).

Let also the sets $\mathcal{X}^f(g, h) = \bigcup_{\tau>0} \mathcal{X}^f(g, h, \tau)$, $\mathcal{X}^b(g, h) = \bigcup_{\tau>0} \mathcal{X}^b(g, h, \tau)$ and $\mathcal{X}(g, h) = \mathcal{X}^f(g, h) \cup \mathcal{X}^b(g, h)$.

Remark 4.1.

(1) The set $\mathcal{X}^f(g, h, \infty)$ ($\mathcal{X}^b(g, h, \infty)$) coincides with the maximal weakly forward(backward) invariant set w.r.t. g contained in the set $\{\xi \in \mathcal{X} : h(\xi) = 0\}$.
  We recall that a subset $K \subset \mathbb{R}^n$ is weakly forward(backward) invariant w.r.t g if for each $\xi \in K$ there exists a solution $\varphi : [0, \infty) \to \mathbb{R}^n$ ($\varphi : (-\infty, 0] \to \mathbb{R}^n$) of $\dot{x} = g(x)$ such that $\varphi(0) = \xi$ and $\varphi(t) \in K$ for all $t \geq 0$ ($t \leq 0$).
(2) If we consider the system with outputs $\dot{x} = g(x)$, $y = h(x)$ and state space $\mathcal{X}$, with $0 \in \mathcal{X}$, $g(0) = 0$ and $h(0) = 0$, then the set $\mathcal{X}^f(g, h)$ coincides with the set of states ξ that cannot be instantaneously distinguished from the zero state through the output y. In the particular case in which g is a linear function, i.e., $g(\xi) = A\xi$ and $h(\xi) = \xi^T C^T C \xi$, and C is a matrix, then $\mathcal{X}(g, h) \subset \mathcal{X} \cap \mathcal{U}$, being $\mathcal{U}$ the unobservable subspace of (C, A).
(3) When g and h are smooth functions we have that
$$\mathcal{X}(g, h) \subset \{\xi \in \mathcal{X} : L_g^k h(\xi) = 0 \, \forall k \in \mathbb{N}_0\},$$
with $L_g^k h$ the k-th. Lie derivative of h along g.

Let us introduce the following assumptions, in order to obtain some convergence criteria based on the invariance results given in Section 3 and on the observability-like notions already introduced. From now on, for a given function $V \in \mathcal{V}$ we considerer the open sets $\mathcal{O}$ and $\mathcal{O}_\gamma$ introduced in Definition 3.4.

Assumption 3. For the forward complete trajectory (x, σ) of (1) there exist a function $V \in \mathcal{V}$ and a family of functions $\{W_\gamma : \mathcal{O}_\gamma \to \mathbb{R}, \gamma \in \Gamma\}$ such that (x, σ) and V satisfy Assumption 1 and in addition

$$-\nabla V_\gamma(\xi) f_\gamma(\xi) \geq W_\gamma(\xi) \geq 0 \quad \forall \xi \in \mathcal{O}_\gamma, \quad \forall \gamma \in \Gamma. \quad (2)$$

Assumption 4. For the forward complete trajectory (x, σ) of (1) there exist a function $V \in \mathcal{V}$ and a family of functions $\{W_\gamma : \mathcal{O}_\gamma \to \mathbb{R}, \gamma \in \Gamma\}$ such that (x, σ) and V satisfy Assumption 2 and in addition (2) holds.

In the sequel we adopt the following notation:
$$\mathcal{O}_{\gamma,\mu}(f, W) := \mathcal{O}_\gamma^b(f_\gamma, W_\gamma) \cap \mathcal{O}_\mu^f(f_\mu, W_\mu).$$

Theorem 6. Let (x, σ) be a forward complete trajectory of (1) with $\sigma \in \mathcal{S}_a$. Then the following holds:

(1) if (x, σ) verifies Assumption 3, then there exists $c \in \mathbb{R}$ such that
$$x \to \bigcup_{\gamma,\gamma' \in \Gamma} \left(\mathcal{O}_{\gamma,\gamma'}(f, W) \cap V_\gamma^{-1}(c) \cap V_{\gamma'}^{-1}(c) \right);$$
(2) if Γ is finite and (x, σ) verifies Assumption 4, then there exists $\boldsymbol{c} \in \mathbb{R}^N$ such that
$$x \to \bigcup_{\gamma,\gamma' \in \Gamma} \left(\mathcal{O}_{\gamma,\gamma'}(f, W) \cap V_\gamma^{-1}(c_\gamma) \cap V_{\gamma'}^{-1}(c_{\gamma'}) \right).$$

Remark 4.2. If in addition to the hypotheses of Theorem 6, we have that for some $x_e \in \cup_{\gamma \in \Gamma} \mathcal{O}_\gamma$, either for all $\gamma \in \Gamma$, $\mathcal{O}_\gamma^f(f_\gamma, W_\gamma) \subset \{x_e\}$ or for all $\gamma \in \Gamma$, $\mathcal{O}_\gamma^b(f_\gamma, W_\gamma) \subset \{x_e\}$, then $x \to x_e$. The first conclusion of Corollary 4.10 in [5] is a particular case of this result.

We note that, according to the particular geometry of each $\mathcal{O}_\gamma$, it could happen that $\mathcal{O}_\gamma^b(f_\gamma, W_\gamma) \neq \mathcal{O}_\gamma^f(f_\gamma, W_\gamma)$ and even that one of those sets be void and the other one not.

In what follows let for each $\gamma \in \Gamma$, $E_\gamma = \{\xi \in \chi_\gamma : f_\gamma(\xi) = 0\}$ the set of equilibrium points of f_γ.

The following convergence result involves an "ergodicity" condition on the switching signals considered.

Theorem 7. Suppose that Γ is a finite set. Let (x, σ), with $\sigma \in \mathcal{S}_e \cap \mathcal{S}_d$, be a forward complete trajectory of (1). Then the following holds:

(1) if (x, σ) verifies Assumption 3 and if for every $\gamma \in \Gamma$, either $\mathcal{O}_\gamma^b(f_\gamma, W_\gamma) = E_\gamma \cap \mathcal{O}_\gamma$ or $\mathcal{O}_\gamma^f(f_\gamma, W_\gamma) = E_\gamma \cap \mathcal{O}_\gamma$, then there exists $c \in \mathbb{R}$ such that $x \to \cap_{\gamma \in \Gamma}(E_\gamma \cap V_\gamma^{-1}(c))$. If, in addition, for each $c \in \mathbb{R}$, $\cap_{\gamma \in \Gamma}(E_\gamma \cap V_\gamma^{-1}(c))$ is a discrete set, then $x \to x_e$ for some $x_e \in \cap_{\gamma \in \Gamma}(E_\gamma \cap \mathcal{O}_\gamma)$.
(2) If (x, σ) verifies Assumption 4 and if for every $\gamma \in \Gamma$, either $\mathcal{O}_\gamma^b(f_\gamma, W_\gamma) = E_\gamma \cap \mathcal{O}_\gamma$ or $\mathcal{O}_\gamma^f(f_\gamma, W_\gamma) = E_\gamma \cap \mathcal{O}_\gamma$, then $x \to \cap_{\gamma \in \Gamma}(E_\gamma \cap \mathcal{O}_\gamma)$. If, in addition, $\cap_{\gamma \in \Gamma}(E_\gamma \cap \mathcal{O}_\gamma)$ is a discrete set, then $x \to x_e$ for some $x_e \in \cap_{\gamma \in \Gamma}(E_\gamma \cap \mathcal{O}_\gamma)$.

In the sequel we give sufficient conditions for the convergence to a given equilibrium point x_e of (1), i.e. a point x_e that verifies $f_\gamma(x_e) = 0$ for all $\gamma \in \Gamma$ such that $x_e \in \chi_\gamma$. We assume, without loss of generality, that x_e is the origin.

Assumption 5. 0 is an equilibrium point of (1).

Assumption 6. For every $\gamma \in \Gamma$ such that $0 \in \chi_\gamma$, the initial value problem $\dot{x} = f_\gamma(x)$, $x(0) = 0$ has a unique solution.

Theorem 8. Suppose that assumptions 5 and 6 hold and let (x, σ) be a forward complete trajectory of (1) with $\sigma \in \mathcal{S}_a$.

(1) If Assumption 3 is verified, $0 \in \mathcal{O}$ and the following holds
  (a) $\mathcal{O}_\gamma^f(f_\gamma, W_\gamma, \infty) \cap \mathcal{O}_\gamma^b(f_\gamma, W_\gamma, \infty) \subset \{0\} \quad \forall \gamma \in \Gamma$,
  (b) $\mathcal{O}_{\gamma,\gamma'}(f, W) \cap V_\gamma^{-1}(c) \cap V_{\gamma'}^{-1}(c) \subset \{0\}$, $\forall \gamma \neq \gamma' \in \Gamma$, $\forall c \in \mathbb{R}$,
  then $x \to 0$.
(2) If Γ is finite, Assumption 4 is verified, $0 \in \mathcal{O}$, and 1.(a) and the following hold
  (a) $\mathcal{O}_{\gamma,\gamma'}(f, W) \subset \{0\}$, $\quad \forall \gamma \neq \gamma' \in \Gamma$,
  then $x \to 0$.

When Γ is finite and σ belongs to $\mathcal{S}_d \cap \mathcal{S}^H$, hypothesis 2. of Theorem 8 can be weakened as follows.

Given a set-valued map $H : \Gamma \rightsquigarrow \Gamma$, a finite sequence $\{\gamma_i\}_{i=1}^m \subset \Gamma$, $m \geq 3$, is a *simple cycle of H* if $\gamma_1 = \gamma_m$,

$\gamma_{i+1} \in H(\gamma_i)$ for all $i = 1, \ldots, m-1$ and if $\gamma_i = \gamma_j$ and $i < j$ then $i = 1$ and $j = m$.

Theorem 9. Suppose that Γ is finite, that $H : \Gamma \rightsquigarrow \Gamma$ and that (x, σ) is a forward complete trajectory of (1) with $\sigma \in \mathcal{S}_d \cap \mathcal{S}^H$. Suppose in addition that assumptions 5 and 6 hold.

(1) If Assumption 4 holds, $0 \in \mathcal{O}$ and
  (a) $\mathcal{O}^f_\gamma(f_\gamma, W_\gamma, \infty) \subset \{0\}$ for every $\gamma \in \Gamma$ or $\mathcal{O}^b_\gamma(f_\gamma, W_\gamma, \infty) \subset \{0\}$ for every $\gamma \in \Gamma$,
  (b) for each simple cycle $\{\gamma_i\}_{i=1}^m$ of H there exists $j \in \{1, \ldots, m-1\}$ such that
$$\mathcal{O}_{\gamma_j,\gamma_{j+1}}(f, W) \subset \{0\}, \tag{3}$$
  then $x \to 0$.
(2) The same conclusion as in 1. holds if we replace Assumption 4 by Assumption 3 and condition 1.(b) by the weaker one:
  (a) for every $c \in \mathbb{R}$ and for each simple cycle $\{\gamma_i\}_{i=1}^m$ of H there exists $j \in \{1, \ldots, m-1\}$ such that
$$\mathcal{O}_{\gamma_j,\gamma_{j+1}}(f, W) \cap V^{-1}_{\gamma_j}(c) \cap V^{-1}_{\gamma_{j+1}}(c) \subset \{0\}. \tag{4}$$

Remark 4.3. It can be seen that Theorem 9 and Theorem 8 (supposing in Part 2. that Assumption 5 holds) remain valid if, instead of Assumption 6, we suppose that the function V in assumptions 3 and 4 verifies the following: for each $\gamma \in \Gamma$ such that $0 \in \chi_\gamma$, $V^{-1}_\gamma(0) \cap \chi_\gamma = \{0\}$. This condition is fulfilled when, for example, $V_\gamma(\cdot)$ is positive definite on χ_γ for every γ such that $0 \in \chi_\gamma$.

4.2 Stability criteria

Combining the convergence results already presented with well known sufficient Lyapunov conditions for the local (global) stability of a family $\mathcal{T}$ of forward complete trajectories of (1), we can derive some new local (global) asymptotic stability criteria.

We recall that a family $\mathcal{T}$ of forward complete trajectories of (1) is

(1) *locally uniformly stable* (LUS) if there exist a positive number $r > 0$ and a function $\alpha : [0, r] \to \mathbb{R}$ of class $\mathcal{K}$ [1] such that for all $(x, \sigma) \in \mathcal{T}$
$$|x(t_0)| \le r \Rightarrow |x(t)| \le \alpha(|x(t_0)|) \quad \forall t \ge t_0, \forall t_0 \ge 0;$$
(2) *globally uniformly stable* (GUS) if there exists a function $\alpha : [0, \infty) \to \mathbb{R}$ of class $\mathcal{K}_\infty$ such that for all $(x, \sigma) \in \mathcal{T}$
$$|x(t)| \le \alpha(|x(t_0)|) \quad \forall t \ge t_0, \forall t_0 \ge 0;$$
(3) *locally asyptotically stable* (LAS) if it is LUS and there exists $\eta > 0$ such that for all $(x, \sigma) \in \mathcal{T}$ with $|x(0)| < \eta$, $x \to 0$;
(4) *globally asyptotically stable* (GAS) is it is GUS and for all $(x, \sigma) \in \mathcal{T}, x \to 0$.

The different stability results that we present next, require the introduction of the following pair of functions.

Definition 4.1. We say that a pair (V, W) is a *weak Lyapunov pair* for the family $\mathcal{T}$ of forward complete trajectories of (1) if

(1) $V \in \mathcal{V}$, $0 \in \mathcal{O}$ and there exist functions α_1 and α_2 of class $\mathcal{K}$ such that
$$\alpha_1(|\xi|) \le V(\xi, \gamma) \le \alpha_2(|\xi|) \quad \forall \xi \in \mathcal{O}_\gamma, \forall \gamma \in \Gamma. \tag{5}$$
(2) $W : \mathrm{dom}(f) \cap (\mathcal{O} \times \Gamma) \to \mathbb{R}$ is such that (2) holds with $W_\gamma(\cdot) = W(\cdot, \gamma)$.
(3) For every $(x, \sigma) \in \mathcal{T}$, the following is verified: $x(t) \in \mathcal{O} \quad \forall t \in [a, b] \subset [0, +\infty) \quad \Rightarrow v(t) = V(x(t), \sigma(t))$ is nonincreasing on $[a, b]$.

We say that a pair (V, W) is a *F-weak Lyapunov pair* for the family $\mathcal{T}$ of forward complete trajectories of (1) if V and W satisfy 1. and 2. and the following condition, which is weaker than 3.

(1) For every $(x, \sigma) \in \mathcal{T}$, the following holds: $x(t) \in \mathcal{O}$ for all $t \in [a, b] \subset [0, +\infty) \Rightarrow$ for every $\gamma \in \Gamma$ $v(t) = V(x(t), \gamma)$ is nonincreasing on $[a, b] \cap \sigma^{-1}(\gamma)$.

By using standard techniques (like those in [2] or in [11]) it is not hard to prove that the existence of a weak Lyapunov pair (or a F-weak Lyapunov pair when Γ is finite) for a family of trajectories $\mathcal{T}$ of (1), implies that $\mathcal{T}$ is LUS and that it is GUS if, in addition, $\mathcal{O} = \mathbb{R}^n$ and V is radially unbounded, i.e. there exist functions α_1 and α_2 of class $\mathcal{K}_\infty$ such that (5) holds.

Theorem 10. Suppose that Assumption 5 holds and let $\mathcal{T}$ be a family of forward complete trajectories of (1) such that for every $(x, \sigma) \in \mathcal{T}$, $\sigma \in \mathcal{S}_a$. Then $\mathcal{T}$ is LAS if one of the following conditions holds:

(1) there exists a weak Lyapunov pair (V, W) for $\mathcal{T}$ such that the restriction of V to $\mathrm{dom}(f) \cap (\mathcal{O} \times \Gamma)$ is continuous and 1.(a) and 1.(b) of Theorem 8 hold.
(2) Γ is finite and there exists a F-weak Lyapunov pair (V, W) for $\mathcal{T}$ such that 1.(a) and 2.(a) of Theorem 8 hold.

If, in addition, $\mathcal{O} = \mathbb{R}^n$ and V is radially unbounded, then $\mathcal{T}$ is GAS.

Theorem 11. Suppose that Γ is finite and that Assumption 5 holds. Let $\mathcal{T}$ be a family of forward complete trajectories of (1) such that for every $(x, \sigma) \in \mathcal{T}$, $\sigma \in \mathcal{S}_d \cap \mathcal{S}^H$, with $H : \Gamma \rightsquigarrow \Gamma$. Then $\mathcal{T}$ is LAS if one of the following holds.

(1) There exists a weak Lyapunov pair (V, W) such that 1.(a) and 2.(a) of Theorem 9 hold.
(2) There exists a F-weak Lyapunov pair (V, W) such that 1.(a) and 1.(b) of Theorem 9 hold.

If, in addition, $\mathcal{O} = \mathbb{R}^n$ and V is radially unbounded, then $\mathcal{T}$ is GAS.

Theorem 12. Suppose that Γ is finite and let $\mathcal{T}$ be a family of forward complete trajectories of (1) such that for every $(x, \sigma) \in \mathcal{T}$, $\sigma \in \mathcal{S}_e \cap \mathcal{S}_d$. Suppose that there exists a F-weak Lyapunov pair (V, W) for $\mathcal{T}$ such that for all $\gamma \in \Gamma$, either $\mathcal{O}^b_\gamma(f_\gamma, W_\gamma) = E_\gamma \cap \mathcal{O}_\gamma$ or $\mathcal{O}^f_\gamma(f_\gamma, W_\gamma) = E_\gamma \cap \mathcal{O}_\gamma$ and that $\cap_{\gamma \in \Gamma} (E_\gamma \cap \mathcal{O}_\gamma) = \{0\}$. Then $\mathcal{T}$ is LAS.

If, in addition, $\mathcal{O} = \mathbb{R}^n$ and V is radially unbounded, then $\mathcal{T}$ is GAS.

Proof of Theorems 10, 11 and 12. Since the hypotheses of the three theorems imply that $\mathcal{T}$ is locally uniformly stable (LUS), we only need to prove that there exists $\eta > 0$ such that for every $(x, \sigma) \in \mathcal{T}$, $|x(0)| < \eta$ implies that $x \to 0$.

[1] As usual, by a $\mathcal{K}$-function we mean a function $\alpha : [0, r] \to \mathbb{R}_{\ge 0}$ that is strictly increasing and continuous, and satisfies $\alpha(0) = 0$. A $\mathcal{K}_\infty$-function is one of class $\mathcal{K}$ for which $r = +\infty$ and that is in addition unbounded.

Since $\mathcal{T}$ is LUS, there exist $\eta > 0$ and $\rho > 0$ such that, for every $(x, \sigma) \in \mathcal{T}$ with $|x(0)| < \eta$, $x(t) \in B = \{\xi \in \mathbb{R}^n : |\xi| \leq \rho\} \subset \mathcal{O}$ for all $t \geq 0$. Therefore $(x, \sigma) \in \mathcal{T}$ is precompact relative to $\mathcal{O}$ whenever $|x(0)| < \eta$. Then, due to Remark 4.3, to Theorem 8 in the case of Theorem 10 and to Theorem 9 in the case of Theorem 11, we have that $x \to 0$ for any $(x, \sigma) \in \mathcal{T}$ such that $|x(0)| < \eta$.

In the case of Theorem 12, due to Theorem 7 we have that for every $(x, \sigma) \in \mathcal{T}$ such that $|x(0)| < \eta, x \to \cap_{\gamma \in \Gamma} (E_\gamma \cap \mathcal{O}_\gamma) = \{0\}$. In consequence the local asymptotic stability of $\mathcal{T}$ follows.

When $\mathcal{O} = \mathbb{R}^n$ and V is radially unbounded, we have that $\mathcal{T}$ is GUS. That $x \to 0$ for every $(x, \sigma) \in \mathcal{T}$ follows by using the fact that any trajectory of $\mathcal{T}$ is precompact and the same arguments as above. ∎

Remark 4.4. Theorem 12 strengthens Theorem 15 in [17] (which is the extension of the main result of [3] to nonlinear switched systems). In fact, the hypotheses of Theorem 12 are weaker than those of that theorem since, on one hand, even when restricted to the case $V(\xi, \gamma) = V(\xi)$ and $\mathcal{O} = \mathbb{R}^n$ (as that theorem considers) the condition $\cap_{\gamma \in \Gamma} E_\gamma = \{0\}$ is weaker than the hypothesis that V is a common joint Lyapunov function as is assumed in that work and, on the other hand, the condition $\mathcal{O}_\gamma^f(f_\gamma, W_\gamma) = E_\gamma$ is weaker than the condition $M \cap Z_\gamma = E_\gamma$ (with $Z_\gamma = \{\xi : W_\gamma(\xi) = 0\}$) considered in [17], since it always holds that $\mathcal{O}_\gamma(f_\gamma, W_\gamma) \subset M \cap Z_\gamma$ and sometimes the inclusion is strict.

From Theorem 12 and Remark 4.1.2. we can easily derive the following result, that contains as a particular case Theorem 1 of [3].

Corollary 13. Assume that Γ is finite and that $f_\gamma(\xi) = A_\gamma \xi$ with $A_\gamma \in \mathbb{R}^{n \times n}$ for all $\xi \in \mathbb{R}^n$. Let $\mathcal{T}$ be a family of forward complete trajectories of (1) such that for all $(x, \sigma) \in \mathcal{T}$, $\sigma \in \mathcal{S}_d \cap \mathcal{S}_e$. Suppose that there exists a family of positive definite matrixes $\{P_\gamma, \gamma \in \Gamma\} \subset \mathbb{R}^{n \times n}$ and a family of matrixes $\{C_\gamma, \gamma \in \Gamma\}$ such that

(1) $P_\gamma A_\gamma + A_\gamma^T P_\gamma \leq -C_\gamma^T C_\gamma$ for all $\gamma \in \Gamma$;
(2) $v(t) = x^T(t) P_{\sigma(t)} x(t)$ is nonincreasing on $[0, \infty)$ for all $(x, \sigma) \in \mathcal{T}$;
(3) for every $\gamma \in \Gamma$, $\mathcal{U}_\gamma$, the unobservable subspace of the pair (C_γ, A_γ), coincides with $\ker(A_\gamma)$;
(4) $\cap_{\gamma \in \Gamma} \ker(A_\gamma) = \{0\}$.

The, $\mathcal{T}$ is GAS.

5. CONCLUSIONS

In this paper we have obtained some invariance results for switched systems which satisfy a dwell-time condition. These results enable us to study, in an unified way, properties of bounded trajectories of switched systems whose switchings are subjected not only to state-dependent constraints, but also to restrictions on the accessibility from each subsystem to other ones.

We also derived from these results some convergence and stability criteria. These criteria involve observability-like conditions on functions which bound the derivatives of some continuous functions that are nonincreasing along complete trajectories of the switched systems.

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