

A TRUNCATED VERSION OF THE BIRNBAUM-SAUNDERS DISTRIBUTION WITH AN APPLICATION IN FINANCIAL RISK

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ABSTRACT

In many Solvency and Basel loss data, there are thresholds or deductibles that affect the analysis capability. On the other hand, the Birnbaum-Saunders model has received great attention during the last two decades and it can be used as a loss distribution. In this paper, we propose a solution to the problem of deductibles using a truncated version of the Birnbaum-Saunders distribution. The probability density function, cumulative distribution function, and moments of this distribution are obtained. In addition, properties regularly used in insurance industry, such as multiplication by a constant (inflation effect) and reciprocal transformation, are discussed. Furthermore, a study of the behavior of the risk rate and of risk measures is carried out. Moreover, estimation aspects are also considered in this work. Finally, an application based on real loss data from a commercial bank is conducted.

KEYWORDS

Deductible; risk analysis; loss models; truncated distributions

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1 INTRODUCTION

The assumption of normality is not often quite true when monetary loss data are analyzed. Thus, standard statistical techniques based on the normal model are not adequate in this case. In order to solve this problem, some solutions have been proposed in the literature related to this issue. These solutions can come from two approaches: (i) to transform the loss data and model these by standard methods (although this approach is not always convincing because the interpretation of the results is sometimes misleading, and it has been shown that a data analysis conducted under a wrong transformation reduces the power of the study); and (ii) to directly model the loss data by an appropriate distribution avoiding their transformation. In this last approach, several probability models have been used to describe this type of data. These models are called loss or life distributions. In particular, the Birnbaum-Saunders (BS), exponential, gamma, inverse Gaussian, lognormal, Pareto, and Weibull models have been considered as loss or life distributions; for more details, see Lawless (2002), Kleiber and Kotz (2003), Marshall and Olkin (2007), and Ahmed et al. (2008), among others. In this paper, we use the second approach considering the following ingredients: (i) the basic objective of the actuarial science to model the behavior of loss data based on financial risk; (ii) the BS model as loss distribution; and (iii) the truncation of distributions with the goal of building a model that can be used to forecast losses.

Next, we discuss the mentioned ingredients and propose an application of the truncated version of the BS distribution to improve a forecasting actuarial model. The first considered ingredient is related to financial risk, which has been applied to different fields and nowadays is being used for important initiatives such as Basel II and Solvency II, for example in the development of calculations of operational and credit risk losses in order to evaluate capital requirement; see Sandstrom (2006), and Hull (2007). Also, financial risk is being used by the insurance industry to provide tools to the insurer to fulfill claims in the future; see Klugman et al. (2004), and Panjer (2006). The second ingredient is the two-parameter BS distribution proposed by Birnbaum and Saunders (1969), which has been a model widely studied and applied later on. This is due to the interesting properties of the BS distribution and its close relationship with the normal model. These aspects make the BS distribution a natural and meaningful alternative candidate to the normal model under positive skewness and non-negative support, such as is the case of loss data. The BS distribution has been regularly used in biological, engineering, environmental, and medical applications; see Sanhueza et al. (2008). We believe that this distribution can be used in actuarial science, business, economics, finance and risk management as well. For more

details about the BS model and some of its generalizations and extensions, see Johnson et al. (1995, p. 651), Saunders (2007), and Sanhueza et al. (2008). The BS distribution is implemented in the R software and available from `CRAN.R-project.org` by the `bs` package; see Leiva et al. (2006), and R Development Core Team (2008). Finally, as mentioned, the third ingredient is based on truncated distributions. Truncated data often appear in many fields such as reliability, actuarial science, and chemometrics (where there are instruments that unable to detect data under or over certain limits). Truncated versions of the gamma, inverse beta, lognormal and normal distributions can be revised in Wingo (1988), McEwen and Parresol (1991), Coffey and Muller (2000), Rigby and Stasinopoulos (2006), Nadarajah (2008), and Pichugina (2008).

The described ingredients allow us to develop an actuarial model that describes the distributional behavior of loss data. Specifically, in this paper, we propose a new model called the truncated Birnbaum-Saunders (TBS) distribution. We consider that a direct application of this distribution can be used by the financial industry, as for example, for modeling data from insurance payments that establish a deductible. Another example is related to loss data from financial institutions considering events that are over a minimum amount. These two examples can be studied by truncated distributions instead of the classical perspective from non-truncated distributions. If we agree with the application in both of the mentioned examples, then the utilization of the TBS distribution can be a useful tool for modelling of data based on Solvency II and Basel II; see Sandstrom (2006).

This paper is structured as follows. In Section 2, we provide a background of the BS distribution and a characterization of the TBS distribution, including their probability density function (pdf), cumulative distribution function (cdf), risk indicators, moments, some properties and transformations, and parameter estimation based on the maximum likelihood (ML) method. In Section 3, we carry out a financial application of the obtained results. In the concluding remark and appendix sections, we include some final comments of the proposed TBS model and the proofs of the theorems, respectively.

2 THE NEW MODEL

In this section, we provide a background about the BS distribution and characterize the TBS distribution. We only consider the case of a distribution truncated to the left since cases of truncation to the right or double truncation are similar. In addition, truncation to the left is coherent with the considered financial application. Thus, from now on, we simply refer as TBS distribution when the BS distribution is truncated to the left.

2.1 Background

If a random variable (rv) X follows a BS distribution with shape and scale parameters, α and β , respectively, then the notation $X \sim \text{BS}(\alpha, \beta)$ is used. Thus, if

$$X = \frac{\beta}{4} \left[\alpha Z + \sqrt{\alpha^2 Z^2 + 4} \right]^2 \sim \text{BS}(\alpha, \beta), \quad (2.1)$$

where $\alpha > 0$ and $\beta > 0$, then $Z = [X/\beta]^{1/2} - [\beta/X]^{1/2} / \alpha \sim \text{N}(0, 1)$ and $W = Z^2 = [X/\beta + \beta/X - 2] / \alpha^2 \sim \chi^2(1)$. Therefore, the pdf and cdf of X are, respectively, given by

$$f_X(x) = \phi \left(\frac{1}{\alpha} \xi \left(\frac{x}{\beta} \right) \right) \frac{1}{\alpha \beta} \xi' \left(\frac{x}{\beta} \right) \quad \text{and} \quad F_X(x) = \Phi \left(\frac{1}{\alpha} \xi \left(\frac{x}{\beta} \right) \right), \quad x > 0, \quad (2.2)$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the pdf and cdf of the standard normal distribution, $\xi(x) = [x^{1/2} - x^{-1/2}]$, and $\xi'(x) = d\xi(x)/dx = [x^{-1/2} + x^{-3/2}]/2$. Some properties of the BS model are $cX \sim \text{BS}(\alpha, c\beta)$, with $c > 0$, and $1/X \sim \text{BS}(\alpha, 1/\beta)$. The q th quantile of X is $x_q = [\beta/4] [\alpha z_q + \sqrt{\alpha^2 z_q^2 + 4}]^2$, with $0 < q < 1$, where z_q is the q th quantile of the standard normal distribution. Thus, since $x_{0.5} = \beta$, then β is also the median of the distribution. The r th moment of X is given by

$$\mathbb{E}[X^r] = \beta^r \sum_{j=0}^r \binom{2r}{2j} \sum_{i=0}^j \binom{j}{i} \frac{(2r-2j+2i)!}{2^{r-j+i}(r-j+i)!} \left[\frac{\alpha}{2} \right]^{2r-2j+2i}, \quad r = 1, 2, \dots \quad (2.3)$$

Based on Equation (2.3), the mean, variance and coefficients of variation (CV), skewness (CS) and kurtosis (CK) are, respectively, given by

$$\begin{aligned} \mathbb{E}[X] &= \frac{\beta}{2} [\alpha^2 + 2], \quad \text{Var}[X] = \frac{\beta^2}{4} [5\alpha^4 + 4\alpha^2], \quad \text{CV}[X] = \frac{\sqrt{5\alpha^4 + 4\alpha^2}}{\alpha^2 + 2}, \\ \text{CS}[X] &= \frac{44\alpha^3 + 24\alpha}{[\sqrt{5\alpha^2 + 4}]^3}, \quad \text{and} \quad \text{CK}[X] = 3 + \frac{558\alpha^4 + 240\alpha^2}{[5\alpha^2 + 4]^2}. \end{aligned} \quad (2.4)$$

2.2 Density and properties of the TBS distribution

If $X \sim \text{BS}(\alpha, \beta)$, then $T \sim \text{TBS}_\kappa(\alpha, \beta)$ denotes the truncated version of X at the positive value κ .

Theorem 1. *Let $T \sim \text{TBS}_\kappa(\alpha, \beta)$. Then, the pdf of T is*

$$f_T(t) = \frac{\phi \left(\frac{1}{\alpha} \xi \left(\frac{t}{\beta} \right) \right) \xi' \left(\frac{t}{\beta} \right)}{\alpha \beta \Phi \left(-\frac{1}{\alpha} \xi \left(\frac{\kappa}{\beta} \right) \right)}, \quad t \geq \kappa > 0,$$

where $\xi(\cdot)$ and $\xi'(\cdot)$ are given in Equation (2.2).

Remark 1. If $T \sim \text{TBS}_\kappa(\alpha, \beta)$, then the rv $Z = \left[\frac{[T/\beta]^{1/2} - [\beta/T]^{1/2}}{\alpha} \right]$ follows a standard normal distribution truncated to the left at η , which is denoted by $Z \sim \text{TN}_\eta(0, 1)$; see Cohen (1991, p. 9). Thus, $T = [\beta/4][\alpha Z + \sqrt{\alpha^2 Z^2 + 1}]^2 \sim \text{TBS}_\kappa(\alpha, \beta)$ and the truncation point is $\kappa = [\beta/4][\alpha\eta + \sqrt{\alpha^2\eta^2 + 4}]^2$. This demonstrates that the TBS distribution may be obtained from two ways: (i) by truncating the BS distribution to the left at κ , or (ii) by generating an rv with BS distribution as in Equation (2.1) from $Z \sim \text{TN}_\eta(0, 1)$, where $\eta = \left[\frac{[\kappa/\beta]^{1/2} - [\beta/\kappa]^{1/2}}{\alpha} \right]$.

Corollary 2.1. Let $T \sim \text{TBS}_\kappa(\alpha, \beta)$. Then, the mode of T , denoted by t_m , is given as the solution of the equation $t_m^3 + [\alpha^2 + 1]\beta t_m^2 + \alpha^2 \beta^2 t_m - 2\alpha^2 \beta^3 = 0$, for $t_m \geq \kappa > 0$.

Remark 2. We note that if the solution of the equation given in Corollary 2.1 does not satisfy the condition $t_m \geq \kappa > 0$, then κ is the mode of T .

In general, the cdf of an rv can be used to compute probabilities and quantiles of the distribution, generate random numbers, and produce goodness-of-fit. Next, we find the cdf of the TBS distribution.

Theorem 2. Let $T \sim \text{TBS}_\kappa(\alpha, \beta)$. Then, the cdf of T is

$$F_T(t) = \frac{\Phi\left(\frac{1}{\alpha}\xi(t/\beta)\right) - \Phi\left(\frac{1}{\alpha}\xi(\kappa/\beta)\right)}{\Phi\left(-\frac{1}{\alpha}\xi(\kappa/\beta)\right)}, \quad t \geq \kappa > 0.$$

Corollary 2.2. Let $T \sim \text{TBS}_\kappa(\alpha, \beta)$. Then, the q th quantile of T is expressed as

$$t_q = F_T^{-1}(q) = \frac{\beta}{4} \left[\alpha z_q + \sqrt{\alpha^2 z_q^2 + 4} \right]^2,$$

where z_q is the q th quantile of the standard normal distribution truncated to the left at $\eta = \left[\frac{[\kappa/\beta]^{1/2} - [\beta/\kappa]^{1/2}}{\alpha} \right]$ and $F_T^{-1}(\cdot)$ is the inverse function of $F_T(\cdot)$.

Some transformations of variates are useful in diverse situations. For example, in actuarial science, multiplication by a constant is equivalent to applying loss size inflation uniformly across all loss levels; see Panjer (2006, p. 84). If the scale transformation of an rv follows the same distributional class, then its distribution belongs to the scale family. If the reciprocal transformation of an rv follows the same distributional class, then its distribution belongs to the closed under reciprocation family; see Saunders (1974). The following theorems show some properties and transformations of the TBS distribution.

Theorem 3. Let $T \sim \text{TBS}_\kappa(\alpha, \beta)$. Then, the following properties hold:

- (i) $cT \sim \text{TBS}_{c\kappa}(\alpha, c\beta)$, with $c > 0$, and
- (ii) $1/T$ follows a BS model with parameters α and $1/\beta$ truncated to the right at $1/\kappa$.

There are several reasons motivating the study of logarithmic transformations of variates; see Balakrishnan et al. (2009). Models associated with these transformations are called logarithmic distributions. For more details about logarithmic distributions; see Marshall and Olkin (2007, pp. 427-450).

Remark 3. The sinh-normal (SN) model is a three-parameter logarithmic distribution with shape, location, and scale parameters, $\alpha > 0$, $\gamma \in \mathbb{R}$, and $\sigma > 0$, respectively. This model is sometimes called the log-Birnbaum-Saunders distribution. This is because if $Y \sim \text{SN}(\alpha, \gamma, \sigma)$ and $\sigma = 2$, then $T = \exp(Y) \sim \text{BS}(\alpha, \beta)$, where $\beta = \exp(\gamma)$. For more details about the SN distribution, see Rieck and Nedelman (1991), Johnson et al. (1995, pp. 645-661), and Leiva et al. (2007).

Theorem 4. *Let $T \sim \text{TBS}_\kappa(\alpha, \beta)$. Then, $\log(T) \sim \text{TSN}_{\log(\kappa)}(\alpha, \log(\beta), 2)$, where this notation means the truncated version to the left at $\log(\kappa)$ of the SN distribution.*

Transformations by raising to a power have been found to provide useful and flexible models; see Kleiber and Kotz (2003, p. 148), and Marshall and Olkin (2007, p. 228).

Theorem 5. *Let $T \sim \text{TBS}_\kappa(\alpha, \beta)$. Then, if $Y = T^r$, with $r > 0$, the pdf of Y is given by*

$$f_Y(y) = \frac{1}{2\alpha\sqrt{2\pi}r\eta^{\frac{1}{2r}}} \frac{\exp\left(-\frac{1}{2\alpha^2} \left[\left[\frac{y}{\eta}\right]^{\frac{1}{r}} + \left[\frac{\eta}{y}\right]^{\frac{1}{r}}\right] - 2\right)}{\Phi\left(-\frac{1}{\alpha}\xi(k/\eta^{1/r})\right)} \frac{[y^{\frac{1}{r}} + \eta^{\frac{1}{r}}]}{y^{\frac{1}{2r}+1}}, \quad y \geq k^r, \eta = \beta^r.$$

2.3 Moments of the TBS distribution

The positive integer moments are quite useful in inference and model fitting, while negative moments are less known and their applications are more limited. For details about applications of moments of any order; see Balakrishnan et al. (2009). The moments of the TBS distribution depend on the moments of the truncated standard normal distribution. Some results related to the moments of the truncated normal distribution are shown next.

Theorem 6. *Let $Z \sim \text{TN}_\eta(0, 1)$. Then, for $r \geq 1$,*

(i)

$$\mathbb{E}[Z^r] = \int_{\eta}^{+\infty} z^r \frac{\phi(z)}{\Phi(-\eta)} dz = \eta^{r-1} Q(\eta) + [r-1] \mathbb{E}[Z^{r-2}],$$

where $Q(\eta) = \mathbb{E}[Z] = \int_{\eta}^{+\infty} \frac{z\phi(z)}{\Phi(-\eta)} dz = \frac{\phi(\eta)}{\Phi(-\eta)}$, and

(ii)

$$\begin{aligned} \mathbb{E} \left[Z^{2r+1} [\alpha^2 Z^2 + 4]^{1/2} \right] &= 2Q(\eta) \sqrt{\alpha^2 \eta^2 + 4} + 2r W_{2r-1} \\ &+ \int_{\left[\frac{\alpha\eta}{2}\right]^2 + 1}^{+\infty} \exp\left(\frac{2}{\alpha^2}\right) \frac{[u-1]^r}{\sqrt{2\pi} u^{1/2}} \exp\left(-\frac{2u}{\alpha^2}\right) \frac{\alpha^2}{2\Phi(-\eta)} du, \end{aligned}$$

$$\text{where } W_r = \mathbb{E} \left[Z^r \left[\left[\frac{\alpha Z}{2} \right]^2 + 1 \right]^{1/2} \right].$$

Theorem 7. Let $T \sim \text{TBS}_{\kappa}(\alpha, \beta)$. Then, for $r \geq 1$, the r th moment of T is given by

$$\begin{aligned} \mathbb{E}[T^r] &= \beta^r \sum_{h=0}^r \binom{2r}{2h} \sum_{j=0}^h \binom{h}{j} \left[\frac{\alpha}{2}\right]^{2r-2h+2j} \mathbb{E} \left[Z^{2r-2h+2j} \right] \\ &+ \sum_{h=0}^{r-1} \binom{2r}{2h+1} \sum_{j=0}^h \binom{h}{j} \left[\frac{\alpha}{2}\right]^{2r-2h+2j-1} \mathbb{E} \left[Z^{2r-2h+2j-1} \left[\left[\frac{\alpha Z}{2} \right]^2 + 1 \right]^{1/2} \right], \end{aligned}$$

where $Z \sim \text{TN}_{\eta}(0, 1)$.

Remark 4. If $T \sim \text{BS}(\alpha, \beta)$, then the expression $W_{2r+1} = \mathbb{E} \left[Z^{2r+1} \left[\left[\frac{\alpha Z}{2} \right]^2 + 1 \right]^{1/2} \right]$ (i.e., the odd moments of Z , where $Z \sim \text{N}(0, 1)$) given in Theorem 7 vanishes. However, in the TBS case, these expressions must be considered since such terms are different from zero.

Corollary 2.3. Let $T \sim \text{TBS}_{\kappa}(\alpha, \beta)$ and $Z \sim \text{NT}_{\eta}(0, 1)$. Then, the mean, variance, CV, CS, and CK of T are, respectively, expressed as

$$\begin{aligned} \mathbb{E}[T] &= \beta \left[\frac{\alpha^2}{2} V_2 + 1 + \alpha W_1 \right], \\ \text{Var}[T] &= \beta^2 \left[\frac{\alpha^4}{2} V_4 + \alpha^2 V_2 - \frac{\alpha^4}{4} V_2^2 - \alpha^3 V_2 W_1 + \frac{\alpha^3}{2} W_3 - \alpha^2 W_1^2 \right], \\ \text{CV}[T] &= \frac{\sqrt{2\alpha^4 V_4 + 4\alpha^2 V_2 - \alpha^4 V_2^2 - 2\alpha^3 V_2 W_1 + 2\alpha^3 W_3 - 4\alpha^2 W_1^2}}{\alpha^2 V_2 + 2 + 2\alpha W_1}, \\ \text{CS}[T] &= -2\alpha^3 \left[\frac{6V_2 W_1 - W_1^3 - 2W_3 + 6V_2^2 \alpha - 6V_4 \alpha - 3V_2 W_1^2 \alpha + 3W_1 W_3 \alpha - 3V_2^2 W_1 \alpha^2}{[-\alpha^2 [W_1^2 + V_2^2 \alpha - 2\alpha [W_3 + V_4 \alpha]]]^{3/2}} \right. \\ &\quad \left. + \frac{3V_4 W_1 \alpha^2 + 3V_2 W_3 \alpha^2 - 2W_5 \alpha^2 - V_2^2 \alpha^3 + 3V_2 V_4 \alpha^3 - 2V_6 \alpha^3}{[-\alpha^2 [W_1^2 + V_2^2 \alpha - 2\alpha [W_3 + V_4 \alpha]]]^{3/2}} \right], \quad \text{and} \end{aligned}$$

$$\begin{aligned}
\text{CK}[T] = & - \left[\frac{-16V_4 - 24V_2W_1^2 + 3W_1^4 - 12\alpha^2V_2^2W_1 + 16W_1W_2 - 48V_2^2W_1\alpha + 48V_4W_1\alpha}{[-4V_2 + W_1^2 + 2\alpha V_2W_1 - 2\alpha W_3 - 2\alpha^2V_4 + \alpha^2V_2^2]^2} \right. \\
& + \frac{12V_2W_1^3\alpha + 16V_2W_3\alpha - 12W_1^2W_3\alpha - 16W_5\alpha - 24V_2^3\alpha^2 + 48V_2V_4\alpha^2}{[-4V_2 + W_1^2 + 2\alpha V_2W_1 - 2\alpha W_3 - 2\alpha^2V_4 + \alpha^2V_2^2]^2} \\
& + \frac{-32V_6\alpha^2 + 18V_2^2W_1^2\alpha^2 - 12V_4W_1^2\alpha^2 - 24V_2W_1W_3\alpha^2 + 16W_1W_5\alpha^2}{[-4V_2 + W_1^2 + 2\alpha V_2W_1 - 2\alpha W_3 - 2\alpha^2V_4 + \alpha^2V_2^2]^2} \\
& + \frac{12V_2^3W_1\alpha^3 - 24V_2V_4W_1\alpha^3 + 16V_6W_1\alpha^3 + 16V_6W_5\alpha^3 + 16V_2W_5\alpha^3}{[-4V_2 + W_1^2 + 2\alpha V_2W_1 - 2\alpha W_3 - 2\alpha^2V_4 + \alpha^2V_2^2]^2} \\
& \left. + \frac{3V_2^4\alpha^4 - 8W_7\alpha^3 - 12V_2^2V_4\alpha^4 + 16V_2V_6\alpha^4 - 8V_8\alpha^4}{[-4V_2 + W_1^2 + 2\alpha V_2W_1 - 2\alpha W_3 - 2\alpha^2V_4 + \alpha^2V_2^2]^2} \right],
\end{aligned}$$

where $V_k = \mathbb{E}[Z^k]$ and W_k are given in Theorem 6 and Remark 4.

2.4 Risk rate of the TBS distribution

An rv T with pdf $f_T(t)$ and cdf $F_T(t)$ that follows a loss distributions is equivalently characterized by the survival function, $S_T(t) = 1 - F_T(t)$, or by the risk rate (also known as failure or hazard rate), $h_T(t) = f_T(t)/S_T(t)$, for $t > 0$ and $0 < S_T(t) < 1$. For example, if $h_T(t)$ is non-decreasing (or non-increasing) in t , then $F_T(t)$ belongs to the class of increasing (or decreasing) failure rate (IFR) (or DRF) distributions; if $h_T(t) = \lambda > 0$, for all t , we have $S_T(t) = \exp(-\lambda t)$, for $t > 0$, so that $F_T(t)$ is the cdf of an exponential distribution. The failure rate average (FRA) of T is given by $\text{FRA}(t) = H_T(t)/t = [\int_0^t h_T(t) dt]/t$, for $t > 0$, where $H_T(t)$ is the cumulative risk rate. An analysis of $\text{FRA}(t)$ on t allows us to obtain the increasing (or decreasing) failure rate average (IFRA or DFRA) classes. The conditional survival function of T is defined as $S_T(t|x) = S_T(t+x)/S_T(x)$, for $t > 0$, $x > 0$, and $0 < S_T(\cdot) < 1$. Similar to $h_T(t)$ and $\text{FRA}(t)$, the distribution of T belongs to the new better than used (NBU), exponential, or new worse than used (NWU) classes, when $S_T(t|x) < S_T(t)$, $S_T(t|x) = S_T(t)$, or $S_T(t|x) > S_T(t)$, respectively. In addition, a study of the behavior of $S_T(t|x)$ establishes the loss families called increasing residual life (IRL) and decreasing residual life (DRL). Based on the definition of $h_T(t)$, we have $h_T(t) = -d \log(S_T(t))/dt$, for $t > 0$, and $0 < S_T(t) < 1$, so that $S_T(t) = \int_0^t \exp(-h_T(x)) dx$. Thus, all the risk indicators mentioned above can be expressed in terms of $h_T(t)$. For this reason, our analysis only focusses on the risk rate.

The normal distribution and its truncated version belong to the IFR family. However, the lognormal model has not an increasing failure rate because it initially increases until reaching its change point and then decreases to zero. The risk rate of the BS model behaves

similarly to the lognormal model, but it decreases until a positive constant value (not at zero). The behavior of the risk rate, with $Y \sim \text{BS}(\alpha, \beta)$, does not depend on the parameter β . Also, for any value of α , we have that this rate is (i) unimodal; (ii) increasing for $y < y_c$, and decreasing for $y > y_c$, where y_c is the change point of $h_Y(y)$; and (iii) approaches $1/[2\alpha^2\beta]$ as $y \rightarrow \infty$. For more details about the BS risk rate, see Kundu et al. (2008).

Theorem 8. Let $T \sim \text{TBS}_\kappa(\alpha, \beta)$. Then, the survival function and risk rate of T are, respectively, given by

$$S_T(t) = \frac{\Phi\left(-\frac{1}{\alpha}\xi(t/\beta)\right)}{\Phi\left(-\frac{1}{\alpha}\xi(\kappa/\beta)\right)} \quad \text{and} \quad h_T(t) = \frac{\phi\left(\frac{1}{\alpha}\xi(t/\beta)\right)\xi'(t/\beta)}{\alpha\beta\Phi\left(-\frac{1}{\alpha}\xi(t/\beta)\right)}, \quad t \geq \kappa > 0.$$

The change point of the risk rate is an important value when the loss distribution does not belong to a family with non-monotone failure rate (IFR and DFR classes). Thus, when this change point is known, the inflection point of the risk is obtained. Next, we study the change point of the TBS risk rate, which we denoted by t_c . As β is a scale parameter, without loss of generality, we consider $\beta = 1$.

Theorem 9. Let $T \sim \text{TBS}_\kappa(\alpha, 1)$. Then, t_c is obtained as the solution of the equation

$$\Phi\left(-\frac{1}{\alpha}\xi(t_c)\right) \left[\alpha^2 \xi''(t_c) - \xi(t_c) [\xi'(t_c)]^2 \right] + \alpha \phi\left(-\frac{1}{\alpha}\xi(t_c)\right) \left[\xi'(t_c) \right]^2 = 0, \quad t_c \geq \kappa > 0,$$

where $\xi''(\cdot)$ is the second derivative of $\xi(\cdot)$.

Remark 5. There is not explicit solution to the equation given in Theorem 9. Thus, a numerical method is required to solve it, with the constrain $t \geq \kappa > 0$. In this case, it is possible that a change point of the risk rate does not exist because the solution of this equation would not necessary satisfy the condition $t \geq \kappa > 0$.

Based on the results obtained for the classical BS distribution, we can establish the following Corollary; see Kundu et al. (2008) and Remark 5.

Corollary 2.4. Let $T \sim \text{TBS}_\kappa(\alpha, \beta)$, $h_T(\cdot)$ be its risk rate, and y_c be the change point of the risk rate of $Y \sim \text{BS}(\alpha, \beta)$, the classical BS distribution with parameters α and β . Then,

- (i) If $y_c \geq \kappa$, $y_c = t_c$ is also the change point of $h_T(t)$, which is an upside-down function, and
- (ii) If $y_c < \kappa$, $h_T(t)$ has no change point and this function monotonically decreases until a positive constant value.

Another important value for the risk rate is its limit behavior. This is because, within the class of loss distributions with non-monotone failure rate (IFR and DFR classes), some risk rates, after reaching its change point, decrease to zero and others become stabilized at a positive constant value, and not at zero.

Theorem 10. Let $T \sim \text{TBS}_\kappa(\alpha, \beta)$ and $h_T(\cdot)$ be its risk rate. Then, $\lim_{t \rightarrow \infty} h_T(t) = 1/[2\alpha^2\beta]$.

2.5 Risk measures for the TBS distribution

Although the risk measures mentioned in Section 2.4 are known and used in practice, financial organizations suffer a high level of exposure, which makes necessary to have efficient risk measures for these organizations with the purpose of mitigating their risks. The financial instruments indicate the degree to which an institution is exposed. Thus, the value at risk (VaR) has become a standard measure to evaluate exposure to the risk. The VaR is a quantile of the risk (or loss) distribution and can also be used in the determination of the amount of capital required to withstand such adverse outcomes. Specifically, let T be a loss rv. Then, the VaR of T at the $100 \times q\%$ level, denoted by $\text{VaR}_q(T)$ or t_q , is the $100 \times q$ quantile of the distribution of T . Thus, if $T \sim \text{TBS}_\kappa(\alpha, \beta)$, based on Corollary 2.2, we have

$$\text{VaR}_q[T] = \frac{\beta}{4} \left[\alpha z_q + \sqrt{\alpha^2 z_q^2 + 4} \right]^2, \quad (2.5)$$

where z_q is the q th quantile of $Z \sim \text{TN}_\eta(0, 1)$, with $\eta = [[\kappa/\beta]^{1/2} - [\beta/\kappa]^{1/2}]/\alpha$.

A risk measure is a mapping from the rv representing the loss associated with the risk to the real line and provides a single number that is intended to quantify the exposure to this risk. The study of these measures has been focused on ensuring their consistence. In this line, Artzner (1999) introduced the concept of coherence of risk measures, which is defined as follows. A risk measure $\rho(\cdot)$ is coherent if, for any two bounded loss variates, say T_1 and T_2 , it has the following properties:

- (i) Subadditivity: $\rho(T_1 + T_2) \leq \rho(T_1) + \rho(T_2)$;
- (ii) Monotonicity: $\rho(T_1) \leq \rho(T_2)$, if $T_1 \leq T_2$;
- (iii) Positive homogeneity: $\rho(cT_1) = c\rho(T_1)$, for all c ; and
- (iv) Translation invariance: $\rho(T_1 + c) = \rho(T_1) + c$, for all c .

In financial risk management, the VaR is used for trading processes over a fixed time period. In this case, the normal distribution is often used for describing gains or losses, in which case the VaR satisfies all coherence properties described above. However, operational and credit risk losses are usually well modelled by loss distributions, i.e., positively skewed distributions with non-negative support, such as those occurs with the BS distribution. However, in this case, the risk measure VaR does not satisfy the first of the four criteria for coherence, i.e., the subadditivity requirement. Since the VaR suffers this undesirable property, a more informative and useful measure, such as the tail value at risk (TVaR), should be used. The TVaR has shown to be a coherent risk measure that does have the property of subadditivity; see Artzner (1999). Therefore, the TVaR seems to be a suitable alternative like operational and credit risk measure. Specifically, let T be a loss rv

and $f_T(\cdot)$, $S_T(\cdot)$, and t_q be its pdf, survival function, and q th quantile, respectively. Then, the TVaR of T at the $100 \times q\%$ level, denoted by $\text{TVaR}_q(T)$, is the expected loss given that the loss exceeds the q th quantile of the distribution of T . That is,

$$\begin{aligned} \text{TVaR}_q[T] = \mathbb{E}[T|T > t_q] &= \frac{\int_{t_q}^{\infty} t f_T(t) dt}{S_T(t_q)} = t_q + \frac{\int_{t_q}^{\infty} [t - t_q] f_T(t) dt}{S_T(t_q)} \\ &= t_q + \frac{\int_{t_q}^{\infty} S_T(t) dt}{S_T(t_q)} = \text{VaR}_q[T] + e(t_q), \end{aligned} \quad (2.6)$$

where $e(t_q) = \int_{t_q}^{\infty} S_T(t) dt / S_T(t_q)$ is known in insurance as the mean excess loss function and in lifetime analysis as mean residual life function; see Panjer (2006, p. 34). From Equation (2.6), we note that TVaR is greater than the corresponding VaR by the average excess of all losses that exceed VaR. A complete review of the risk measure TVaR can be found in Wirch (1999), Tasche (2002), and Acerbi and Tasche (2002). Thus, if $T \sim \text{TBS}_{\kappa}(\alpha, \beta)$, from Equations (2.5) and (2.6), we have

$$\text{TVaR}_q[T] = \frac{\beta}{4} \left[\alpha z_q + \sqrt{\alpha^2 z_q^2 + 4} \right]^2 + \frac{1}{\Phi\left(-\frac{1}{\alpha} \xi(t_q/\beta)\right)} \int_{t_q}^{\infty} \Phi\left(-\frac{1}{\alpha} \xi\left(\frac{t}{\beta}\right)\right) dt, \quad (2.7)$$

where z_q is the q th quantile of $Z \sim \text{TN}_{\eta}(0, 1)$, with $\eta = [(\kappa/\beta)^{1/2} - (\beta/\kappa)^{1/2}] / \alpha$.

2.6 Estimation and inference in the TBS distribution

The log-likelihood function for a random sample $T = [T_1, \dots, T_n]^{\top}$, with $T_i \sim \text{TBS}_{\kappa}(\alpha, \beta)$, for $i = 1, \dots, n$, is $\ell(\theta) = \sum_{i=1}^n \ell_i(\theta)$, where $\theta = [\alpha, \beta]^{\top}$,

$$\ell_i(\theta) = -\log(\Phi(-H)) + \log\left(\phi\left(\frac{1}{\alpha} \xi\left(\frac{t_i}{\beta}\right)\right)\right) + \log\left(\xi'\left(\frac{t_i}{\beta}\right)\right) - \log(\beta) - \log(\alpha), \quad (2.8)$$

and $H = \xi(\kappa/\beta)/\alpha$. The ML estimator of the parameter θ is the solution to the equation $\dot{L}_{\theta} = 0$, where $\dot{L}_{\theta} = [\dot{L}_{\alpha}, \dot{L}_{\beta}]^{\top}$ is the score vector with first derivatives given by

$$\dot{L}_{\alpha} = -\frac{n}{\alpha} + \frac{n}{\alpha} \frac{\phi(H)}{\Phi(H)} H + \frac{ns}{\alpha^3 \beta} + \frac{n\beta}{\alpha^3 r} - \frac{2n}{\alpha^3} \quad \text{and} \quad (2.9)$$

$$\dot{L}_{\beta} = -\frac{n}{\beta} - \frac{n}{\alpha} \xi'\left(\frac{\kappa}{\beta}\right) \frac{\kappa}{\beta^2} \frac{\phi(H)}{\Phi(H)} + \frac{n}{2\alpha^2 \beta} \left[\frac{s}{\beta} - \frac{\beta}{r} \right] + \frac{n}{K(\beta)}, \quad (2.10)$$

with $s = \sum_{i=1}^n t_i / n$, $r = n / [\sum_{i=1}^n 1/t_i]$, and $K(x) = n / [\sum_{i=1}^n 1/[x + t_i]]$, for $x > 0$. These likelihood equations do not present analytical solutions so that the use of numerical iterative methods is necessary. As starting values for this numerical procedure, we suggest using the ML estimates of α and β of the BS distribution; see Leiva et al. (2006).

The asymptotic inference for the parameter vector θ can be based on the normal approximation of its ML estimator, denoted by $\hat{\theta}$. This is $\hat{\theta} \sim N_2(\theta, \Sigma_{\hat{\theta}})$, where $\Sigma_{\hat{\theta}}$ corresponds to the asymptotic variance-covariance matrix for $\hat{\theta}$, which may be approximated by $-\ddot{L}_{\theta}^{-1}$. Here $-\ddot{L}_{\theta}$ is the observed information matrix, which is obtained from the Hessian matrix with second derivatives \ddot{L}_{θ} whose elements are given by

$$\begin{aligned}\ddot{L}_{\alpha\alpha} &= \frac{n}{\alpha^2} - \frac{2n}{\alpha^3} \frac{\phi(H)}{\Phi(H)} \xi \left(\frac{\kappa}{\beta} \right) + \frac{n}{\alpha^5} \xi^3 \left(\frac{\kappa}{\beta} \right) \frac{\phi(H)}{\Phi(H)}, \\ &\quad + \frac{n}{\alpha^4} \xi^2 \left(\frac{\kappa}{\beta} \right) \left[\frac{\phi(H)}{\Phi(H)} \right]^2 - \frac{3ns}{\alpha^4 \beta} - \frac{3n\beta}{\alpha^4 r} + \frac{6n}{\alpha^4}, \\ \ddot{L}_{\alpha\beta} &= \ddot{L}_{\beta\alpha} = -\frac{n\kappa}{\alpha^2 \beta^2} \frac{\phi(H)}{\Phi(H)} \xi' \left(\frac{\kappa}{\beta} \right) + \frac{n\kappa}{2\alpha^4 \beta^2} \xi \left(\frac{\kappa}{\beta} \right) \frac{\phi(H)}{\Phi(H)} \\ &\quad - \frac{n}{2\kappa\alpha^4} \xi \left(\frac{\kappa}{\beta} \right) \frac{\phi(H)}{\Phi(H)} - \frac{n\kappa}{\alpha^3 \beta^2} \xi' \left(\frac{\kappa}{\beta} \right) \left[\frac{\phi(H)}{\Phi(H)} \right]^2 - \frac{ns}{\alpha^3 \beta^2} + \frac{n}{\alpha^3 r}, \quad \text{and} \\ \ddot{L}_{\beta\beta} &= \frac{n}{2\beta^2} + \frac{2nk}{\alpha\beta^3} \frac{\phi(H)}{\Phi(H)} \xi' \left(\frac{\kappa}{\beta} \right) + \frac{n[1-\kappa^2]}{2\kappa\alpha^3 \beta^4} \frac{\phi(H)}{\Phi(H)} + \frac{n\kappa^2}{\alpha^2 \beta^4} \left[\xi' \left(\frac{\kappa}{\beta} \right) \right]^2 \left[\frac{\phi(H)}{\Phi(H)} \right]^2 \\ &\quad + \frac{nk^2}{\alpha\beta^4} \frac{\phi(H)}{\Phi(H)} \xi'' \left(\frac{\kappa}{\beta} \right) - K(\beta) - \frac{ns}{\alpha^2 \beta^3},\end{aligned}$$

where s , r , H , and $K(\cdot)$ are given in Equation (2.10).

For the BS distribution, it is known that $\mathbb{E}[\ddot{L}_{\alpha\beta}] = 0$; see Sanhueza et al. (2008). However, for the TBS model, it can be proved by using Theorem 7 that $\mathbb{E}[\ddot{L}_{\alpha\beta}] \neq 0$. An approximate $100[1 - \zeta]\%$ confidence region, with $0 < \zeta < 1$, for θ is given by $\mathcal{R} = \{\theta \in \mathbb{R}^2 : [\hat{\theta} - \theta]^\top \hat{\Sigma}_{\hat{\theta}}^{-1} [\hat{\theta} - \theta] \leq \chi_{1-\zeta}^2(2)\}$, where, as mentioned, $\hat{\Sigma}_{\hat{\theta}}$ may be approximated by $-\ddot{L}_{\theta}^{-1}$, but now evaluated at $\hat{\theta}$, and $\chi_{1-\zeta}^2(2)$ denotes the $(1 - \zeta)$ th quantile of the χ^2 distribution with two degrees of freedom.

Remark 6. Based on the invariance property of the ML estimators, we can estimate different functions of the parameter θ , for instance, the VaR and TVaR.

3 APPLICATION

In this section, we apply the obtained results to a real data set corresponding to unpaid credits (losses) provided by a commercial bank of the Mexican United States. We analyze the data considering the BS and the TBS distributions. For the TBS distribution, this analysis is performed through an R package developed by the authors called `tbs`, which is available upon request. The `tbs` package contains probabilistic, statistical, and model checking tools.

Specifically, this package has incorporated the pdf, cdf, quantiles, risk rate, survival function, a random number generator, moments, VaR, TVaR, ML estimation, Schwarz (SIC) information criterion, probability-probability (PP) and quantile-quantile (QQ) plots, and the Kolmogorov-Smirnov test for the TBS distribution. The estimations obtained from the `tbs` package are relevant for the calculation of the usual measurements of expected value and volatility, as well as for the calculation of the typical risk measurements that we present in this paper. As a result of the analysis of the VaR, it proves to be true that there is a better way to define exposure to risk, such as the TVaR.

3.1 The problem upon analysis

One of the major problem for a Mexican retail bank is to define the threshold used for prosecuting a default on a credit portfolio. The modelling of loss and amount is fundamental for establishing indicators such as VaR or TVaR and determining the placement cost of new credits in a competitive market. We show how the truncation of the considered distribution allows us a much better management of the credits and when to proceed on a default or just consider it as an expected loss on the business operation. The values of VaR or TVaR seem to be higher when we use the classical BS distribution instead of the TBS model, since events with a greater exposure are more disperse and so the 99th quantile is greater. The use of all the loss data leads to an inadequate evaluation of the real expected loss under the application of a threshold policy. The reduction of the exposure and, as consequence, of the expected loss, is directly related to the efficiency of the whole collection process and the early warning indicators that raise the attention of the management on the credit deterioration process.

3.2 Exploratory data analysis

As mentioned, we consider a data set provided by a commercial bank of the Mexican United States, which represents the unpaid credits (in Mexican pesos \times \$100) that have been passed to portfolio overcome in October, 2007. Table 1 presents a descriptive summary for the loss data. The exploratory data analysis indicates a positive skewness ($CS = 2.49$) with high kurtosis ($CK = 9.48$). The great flexibility of the BS distribution allows us to believe that, for these data, this model can be a good candidate, since it can describe the high values of skewness and kurtosis found in the data. The descriptive analysis also reflects that the 50% of the unpaid credits are less than \$2,531.27. In our study, we must also take into account the range and the volatility of these credits.

Table 1: descriptive statistics for non-truncated (NT) and truncated (T) losses (in Mex\$ $\times 100$)

Loss	Mean	Median	SD	CV	CS	CK	Range	Min.	Max.	n
NT	15385.01	2531.27	26177.97	1.70	2.49	9.48	147605.93	50.22	147656.15	1040
T	19196.91	6515.37	28001.29	1.46	2.17	7.74	147392.03	264.12	147656.15	832

Many financial institutions face the problem of having a large number of losses with small amounts of loss. For the financial institution, the potential cost of a judicial follow-up to recover the credit default is, in many cases, greater than the value of the debt of credit. Generally, the financial institution determines a threshold from which it becomes effective on judicial recovery. Now, according to the financial policy previously exposed, we introduce a threshold at $\kappa = 259.55$ on total unpaid credits. As a result of this truncation procedure, we obtain 208 unpaid credits outside of the analysis. The remaining 832 unpaid credits represent a total of \$1,597 millions of Mexican pesos, about 99% of the total debt defaults by such claims. Table 1 also presents a descriptive summary for truncated unpaid credits. The exploratory data analysis indicates a positive skewness ($CS = 2.17$) with a high kurtosis ($CK = 7.73$). The information delivered by the standard deviation, median, range and extreme values suggests that a heavy-tailed distribution should produce a good fit to the data. We propose the TBS distribution based on the exposed properties.

3.3 Model checking

To check the goodness-of-fit of the BS and TBS models to the non-truncated and truncated loss data, respectively, we produce QQ plots with confidence bands based on these distributions; see Atkinson (1985). These graphical plots are shown in Figure 1 and indicate the adequacy of the BS and TBS distributions to the data.

Remark 7. From Figure 1, distributions with heavier tails than those of the BS and TBS models might be more appropriate to the loss data. We are considering to incorporate generalizations of the TBS distribution that produce heavy tails and other aspects related to the TBS distribution for a future work; see Sanhueza et al. (2008).

3.4 Confirmatory data analysis

The capital requirement can be expressed as the difference between a risk measure (VaR or TVaR) and the best estimate of the technical provision, usually estimated by the expected value of the distribution. As mentioned, the measures of risk VaR and TVaR under the BS and TBS distributions are implemented in the `tbs` package. Thus, once the ML estimates

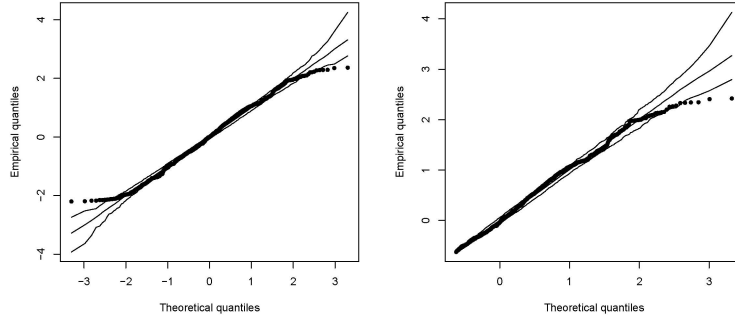


Figure 1: QQ plot with bands based on the BS (left) and TBS (right) models for loss data.

of α and β are obtained, we can estimate the VaR, TVaR, and their capitals by using the invariance property of the ML estimators. These values are presented in Table 2.

Table 2: values for Var, TVaR, and capital for losses (in Mex\$ \times 100).

Distribution	VaR	TVaR	Capital(VaR)	Capital(TVaR)
BS	143255.8	188937.7	127943.7	173625.6
TBS	149919.1	194186.1	85659.5	129926.5

From the exploratory analysis, the nature of the data indicates that the loss distribution would be positively skewed with non-negative support and heavy tails. In addition, a left truncation effect should be considered due to financial decisions. Thus, we must use an appropriate loss distribution truncated to the left, being a good alternative the TBS model. By using goodness-of-fit tools, we have verified that the TBS distribution is a suitable model to the loss data. In order to carry out a financial risk analysis, we must consider that the risk measure VaR is not the most adequate due to its shortage of coherence. Therefore, as mentioned, a good alternative is to use the TVaR because this is a coherent measure. Thus, the TVaR of the TBS distribution is a reasonable risk measure for computing the capital requirement, implying a reduction of this capital when the BST model is the appropriate distribution, as we can see in Table 2.

CONCLUDING REMARKS

In this article, we dealt with a truncated version of the Birnbaum-Saunders distribution. First, we showed that the truncated BS model is more appropriate than the corresponding classical BS model for describing financial data. Second, we characterized and carried out

a risk analysis for the new distribution. Third, we discussed and computed financial risk measures based on the truncated BS model. Specifically, we obtained the pdf, cdf, and moments and presented several properties of the truncated BS distribution. In addition, we considered some transformations associated with the new model. Furthermore, we calculated the change point and studied the limit conduct of the truncated BS risk rate. Finally, we illustrated the presented methodology in this work using an example with real financial data. The ML method and a computational implementation in the R software language have been considered to estimate the parameters of the truncated BS distribution. These estimates were used to compute the usual financial risk measures. This example showed the utility of the new model.

APPENDIX: PROOFS OF THE THEOREMS

Proof of Theorem 1. It is easily obtained from the definition of a truncated distribution and the pdf and cdf of a BS distribution. ■

Proof of Theorem 2. It is obtained from the definition of the cdf of a truncated distribution and the cdf of the BS distribution. ■

Proof of Theorem 3. The proofs of (i) and (ii) are direct from the theorem of change of variable. ■

Proof of Theorem 4. Let $Y = \log(T)$. Then, its pdf is given by

$$f_Y(y) = \frac{1}{\Phi\left(-\frac{1}{\alpha}\xi\left(\frac{k}{\beta}\right)\right)\sqrt{\pi}} \exp\left(-\frac{1}{2\alpha^2}\left[\frac{\exp(y)}{\beta} + \frac{\beta}{\exp(y)} - 2\right]\right) \frac{\exp\left(-\frac{y}{2}\right)[\exp(y) + \beta]}{2\alpha\sqrt{\beta}},$$

for $y > \log \kappa$, which can be written as

$$f_Y(y) = \frac{1}{\Phi\left(-\frac{1}{\alpha}\xi\left(\frac{k}{\beta}\right)\right)} \phi\left(\frac{2}{\alpha} \sinh\left(\frac{y - \log \beta}{2}\right)\right) \frac{1}{\alpha} \cosh\left(\frac{y - \log \beta}{2}\right). \quad (3.1)$$

The last expression given in Equation (3.1) is the pdf of $Y \sim \text{SN}(\alpha, \log(\beta), 2)$, whose cdf is $F_Y(y) = \Phi(2 \sinh([y - \log(\beta)]/2)/\alpha)$ and thus $\Phi(-\xi(k/\beta)/\alpha) = \Phi(-2 \sinh([\log(\kappa) - \log(\beta)]/2)/\alpha)$. ■

Proof of Theorem 5. Considering $Y = T^r$, we obtain $f_Y(y) = f_T(y^{1/r})y^{1/r-1}/r$ if $y^{1/r} \geq k$, where $f_T(\cdot)$ is the pdf of $T \sim \text{TBS}_\kappa(\alpha, \beta)$. Taking $\eta = \beta^r$ and after a few algebraic manipulations, the theorem is proven. ■

Proof of Theorem 6. The expectations in (i) are obtained integrating by the method of parts and recursively. To prove (ii), we use induction. First, we have

$$\mathbb{E}\left[Z\left[1 + \left[\frac{\alpha Z}{2}\right]^2\right]^{1/2}\right] = \mathcal{Q}(\eta)\sqrt{1 + \frac{\alpha^2\eta^2}{4}} + \frac{2\exp\left(\frac{2}{\alpha^2}\right)}{\Phi(-\eta)}\Phi\left(-\frac{2}{\alpha}\sqrt{1 + \frac{\alpha^2\eta^2}{4}}\right),$$

which is obtained by substitution and integration by parts. Considering $\mathbb{E}[Z^{2r+1} [1 + \alpha^2 Z^2/4]^{1/2}] = W_{2r+1}$, for $r \geq 1$, we have $W_{2r+1} = \int_{\eta}^{+\infty} z^{2r+1} \sqrt{1 + \alpha^2 Z^2/4} \exp(-z^2/2) / [\sqrt{2\pi} \Phi(-\eta)] dz$. Now, by using substitution $\alpha^2 Z^2/4 + 1 = u$ and then integration by parts, we reach

$$W_{2r+1} = 2Q(\eta) \sqrt{\frac{\alpha^2 \eta^2}{4} + 1} + 2r W_{2r-1} + \int_{\frac{\alpha^2 \eta^2}{4} + 1}^{+\infty} \frac{\exp\left(\frac{2}{\alpha^2}\right)}{2} \frac{[u-1]^r}{\sqrt{2\pi u^{1/2}}} \frac{\alpha^2 \exp\left(-\frac{2u}{\alpha^2}\right)}{2\Phi(-\eta)} du.$$

To solve the last integral, we use the substitution $2u/\alpha^2 = y^2/2$ and the binomial theorem, such that

$$\int_{\frac{\alpha^2 \eta^2}{4} + 1}^{+\infty} \frac{\exp\left(\frac{2}{\alpha^2}\right)}{2} \frac{[u-1]^r}{\sqrt{2\pi u^{1/2}}} \frac{\alpha^2 \exp\left(-\frac{2u}{\alpha^2}\right)}{2\Phi(-\eta)} du = \frac{\alpha^3}{4} \exp\left(\frac{2}{\alpha^2}\right) \sum_{i=1}^r \binom{r}{i} [-1]^{r-i} \left[\frac{\alpha^2}{4}\right]^i I(i),$$

where, for $i \geq 0$,

$$I(i) = \int_{\frac{2}{\alpha} \sqrt{\frac{\alpha^2 \eta^2}{4} + 1}}^{+\infty} y^{2i} \exp\left(\frac{2}{\alpha^2}\right) \frac{dy}{\sqrt{2\pi} \Phi(-\eta)} \quad \text{and} \quad I(0) = \frac{1 - \Phi\left(\frac{2}{\alpha} \sqrt{\frac{\alpha^2 \eta^2}{4} + 1}\right)}{\Phi(-\eta)}.$$

Thus, recursively, for $i \geq 1$, we have

$$\begin{aligned} I(i) &= \left[\frac{2}{\alpha} \sqrt{\frac{\alpha^2 \eta^2}{4} + 1} \right]^{2i-1} \exp\left(\frac{2}{\alpha^2}\right) Q(\eta) + 2i I(i-1) \\ &= \left[\frac{2}{\alpha} \sqrt{\frac{\alpha^2 \eta^2}{4} + 1} \right]^{2i-1} \exp\left(\frac{2}{\alpha^2}\right) Q(\eta) + 2i! \frac{\left[\Phi\left(-\frac{2}{\alpha} \sqrt{\frac{\alpha^2 \eta^2}{4} + 1}\right) \right]}{\Phi(-\eta)} \\ &\quad + \sum_{j=2}^i \left[\prod_{h=2}^j [2i - (h-2)] \right] \left[\frac{2}{\alpha} \sqrt{\left[\frac{\alpha\eta}{2}\right]^2 + 1} \right]^{2i-j} \exp\left(\frac{2}{\alpha^2}\right) Q(\eta). \end{aligned}$$

Finally,

$$\begin{aligned} W_{2r+1} &= 2Q(\eta) \sqrt{\frac{\alpha^2 \eta^2}{4} + 1} + 2r W_{2r-1} + \sum_{i=1}^r \binom{r}{i} \frac{\alpha^4 [-1]^{r-i}}{8} \left[\sqrt{\frac{\alpha^2 \eta^2}{4} + 1} \right]^{2i-1} Q(\eta) \\ &\quad + \sum_{i=1}^r \binom{r}{i} [-1]^{r-i} \alpha \left[\frac{\alpha^2}{4} \right]^{i+1} \sum_{j=2}^i \left[\prod_{h=2}^j [2i - h + 2] \right] \left[\frac{2}{\alpha} \sqrt{\frac{\alpha^2 \eta^2}{4} + 1} \right]^{2i-j} Q(\eta) \\ &\quad + \exp\left[\frac{2}{\alpha^2}\right] \sum_{i=1}^r \binom{r}{i} [-1]^{r-i} \alpha \left[\frac{\alpha^2}{4} \right]^{i+1} 2i! \frac{\left[1 - \Phi\left(\frac{2}{\alpha} \sqrt{\frac{\alpha^2 \eta^2}{4} + 1}\right) \right]}{\Phi(-\eta)}, \quad r \geq 1. \end{aligned}$$

We note all the expectations exist and can be obtained with no need for numerical methods. ■

Proof of Theorem 7. From Remark 1, taking expectation and using the binomial theorem, we obtain

$$\begin{aligned}
\mathbb{E}[T^r] &= \beta^r \sum_{l=0}^r \binom{r}{l} \mathbb{E} \left[\left[\frac{\alpha^2}{2} Z^2 + \alpha Z \sqrt{\left[\frac{\alpha}{2} Z \right]^2 + 1} \right]^l \right] \\
&= \beta^r \sum_{h=0}^r \binom{2r}{2h} \sum_{j=0}^h \binom{h}{j} \left[\frac{\alpha}{2} \right]^{2[r-h+j]} \mathbb{E} \left[Z^{2[r-h+j]} \right] + \\
&\quad \beta^r \sum_{h=0}^{r-1} \binom{2r}{2h+1} \sum_{j=0}^h \binom{h}{j} \left[\frac{\alpha}{2} \right]^{2[r-h+j]-1} \frac{\mathbb{E} \left[Z^{2[r-h+j]-1} \left[\alpha^2 Z^2 + 4 \right]^{1/2} \right]}{2}. \blacksquare
\end{aligned}$$

Proof of Theorem 8. The expressions are obtained from the definition of the risk rate, and from Theorems 1 and 2. We note that the risk rate of the TBS distribution is defined as that of the BS risk rate, but its domain is defined for $t \geq \kappa > 0$, i.e., its risk rate depends on κ only in its domain. ■

Proof of Theorem 9. It is obtained by the derivative of the risk rate with respect to t for $\beta = 1$. ■

Proof of Theorem 10. The limit of the TBS risk rate does not depend on κ , therefore this result coincides with the limit of the risk rate of the classical BS. ■

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