A characterization of iISS for time-varying impulsive systems

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***Abstract*—Most of the existing characterizations of the integral input-to-state stability (iISS) property are not suitable for time-varying or switched (nonlinear) systems. Previous work by the authors has shown that in such cases where converse Lyapunov theorems for stability are not available, iISS-Lyapunov functions may not exist. In these cases, the iISS property can still be characterized as the combination of global uniform asymptotic stability under zero input (0-GUAS) and uniformly bounded energy input-bounded state (UBEBS). This paper shows that such a characterization remains valid for time-varying impulsive systems, under an appropriate condition on the number of impulse times on each finite time interval.**

***Index Terms*—Stability, impulsive systems, bounded energy, nonlinear systems.**

I. Introduction

Input-to-state stability (ISS) [1] and integral-ISS (iISS) [2] are arguably the most important and useful state-space based nonlinear notions of stability for systems with inputs. The iISS property gives a state bound that is the sum of a decaying-to-zero term whose amplitude depends only on the initial state, and a term depending (nonlinearly) only on an integral of a nonlinear function of the input. The latter term can be interpreted as an input energy bound.

For time-invariant systems described by ordinary differential equations, several characterizations of the iISS property exist (see [2]–[4]). Among the different characterizations, perhaps the most practical ones are those based on iISS-Lyapunov functions [3]. Indeed, since iISS is known to be equivalent to the existence of an iISS-Lyapunov function, there is no loss of generality in focusing on the obtention of such functions. Results that ensure that an iISS system admits the corresponding type of Lyapunov function heavily rely on converse Lyapunov theorems for stability [5], since iISS implies global asymptotic stability.

As for time-varying systems, although some Lyapunov characterizations of ISS exist when the function f defining the system dynamics $\dot{x} = f(t, x, u)$ is continuous [6]–[8], no corresponding Lyapunov characterizations of iISS exist. Moreover, previous work by the authors [9] has shown that if f is not continuous with respect to the time variable, as is the case for switched systems, then iISS-Lyapunov functions may not exist. Also in [9], it is shown that for a wide variety of dynamical systems (where f does not have to be continuous and solutions are not necessarily unique), iISS is equivalent to global uniform asymptotic stability under zero input (0-GUAS) in combination with uniformly bounded energy input-bounded state (UBEBS) (see Section II-B for the precise definitions). Note that although f may be not continuous, the state trajectory is indeed continuous by virtue of being the solution to an ordinary differential equation.

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Impulsive systems are dynamical systems whose state evolves continuously most of the time but may exhibit jumps (discontinuities) at isolated time instants (see [10]). The continuous evolution of the state (i.e. between jumps) is governed by ordinary differential equations. The time instants when jumps occur are part of the impulsive system definition and the after-jump value of the state vector is governed by a static (i.e. not differential) equation. The aim of this paper is to analyze whether and in what ways the previously derived characterization of iISS (namely, iISS = 0-GUAS + UBEBS) can be extended to impulsive systems [11], especially to impulsive systems where both the ordinary differential equations defining continuous state evolution and the static equation defining after-jump values can be time-varying and lack time continuity.

Notation. $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ denote the natural numbers, reals, positive reals and nonnegative reals, respectively. $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^p$. We write $\alpha \in \mathcal{K}$ if $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous, strictly increasing and $\alpha(0) = 0$, and $\alpha \in \mathcal{K}_\infty$ if, in addition, α is unbounded. We write $\beta \in \mathcal{KL}$ if $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $\beta(\cdot, t) \in \mathcal{K}_\infty$ for any $t \geq 0$ and, for any fixed $r \geq 0$, $\beta(r, t)$ monotonically decreases to zero as $t \to \infty$. For every $n \in \mathbb{N}$ and $r \geq 0$, we define the closed ball $B_r^n := \{x \in \mathbb{R}^n : |x| \leq r\}$.

II. Problem Statement

A. Impulsive systems

Consider the time-varying impulsive system with inputs

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad \text{for } t \notin \sigma, \tag{1a}$$

$$x(t) = x(t^-) + g(t, x(t^-), u(t)), \qquad \text{for } t \in \sigma, \tag{1b}$$

where $t_0 \geq 0$ is the initial time, $\sigma = \{\tau_k\}_{k=1}^N$, with N finite or $N = \infty$, is a strictly increasing sequence of impulse times in $\mathbb{R}_{>0}$, the state variable $x(t) \in \mathbb{R}^n$, the continuous-time input variable $u(t) \in \mathbb{R}^m$ and f and g are functions from $\mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$. The ordinary differential equation (1a) defines the continuous evolution of the state vector x and (1b) defines the value of x at the impulse times. To ensure that the jumps in x caused by (1b) cannot occur infinitely frequently, it is assumed that $\tau_k \to \infty$ when $N = \infty$. By convention we define $\tau_0 = 0$ (however, τ_0 is not considered an impulse time) and, when N is finite, we set $\tau_{N+1} := \infty$. We will employ $\mathcal{I}$ to denote the set of all these admissible impulse time sequences, i.e. $\mathcal{I}$ denotes the set of all strictly increasing sequences of positive real numbers that either have a finite number of elements or are unbounded. Let $\mathcal{U}$ be the set of all the functions $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ that are Lebesgue measurable and locally bounded. We will use the term "input" to refer to a pair $w = (u, \sigma) \in \mathcal{U} \times \mathcal{I}$ consisting of a continuous-time input u and an admissible impulse time sequence σ.

Besides the fact that the impulsive system (1) is time-varying (since f and g have explicit time dependence), another difference with respect to standard formulations for impulsive systems with inputs (such as [11]) is that in (1b) the value of x at an impulse time $t \in \sigma$ depends on the instantaneous value $u(t)$ and not on the left limit $u(t^-)$. Hence, we do not need u to have a left limit at any $t \in \sigma$.

A solution to (1) corresponding to an initial time t_0, an initial state $x_0 \in \mathbb{R}^n$ and an input $w = (u, \sigma) \in \mathcal{U} \times \mathcal{I}$ is a right-continuous function $x : [t_0, T_x) \to \mathbb{R}^n$ such that:

i) $x(t_0) = x_0$;
ii) x is a Carathéodory solution of the differential equation $\dot{x}(t) = f(t, x(t), u(t))$ on $[\tau_k, \tau_{k+1}) \cap [t_0, T_x)$ for all $0 \leq k \leq N$; and
iii) for all $t \in \sigma \cap (t_0, T_x)$ it happens that $x(t) = x(t^-) + g(t, x(t^-), u(t))$, where $x(t^-) := \lim_{s \to t^-} x(s)$.

The solution x is said to be maximally defined if no other solution y satisfies $y(t) = x(t)$ for all $t \in [t_0, T_x)$ and has $T_y > T_x$. A solution x is forward complete if $T_x = \infty$. We will use $\mathcal{T}(t_0, x_0, w)$ to denote the set of maximally defined solutions of (1) corresponding to initial time t_0, initial state x_0, and input w. We say that (1) is forward complete for a given $\sigma \in \mathcal{I}$ if for every $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ and $w = (u, \sigma)$ with $u \in \mathcal{U}$, any solution $x \in \mathcal{T}(t_0, x_0, w)$ is forward complete. Given $\sigma \in \mathcal{I}$, we define $n^\sigma_{(t_0,t]}$ to be the number of elements of σ (i.e. the number of jumps) that lie in the interval $(t_0, t]$:

$$n^\sigma_{(t_0,t]} := \#\big[\sigma \cap (t_0, t]\big]. \tag{2}$$

B. Stability definitions

Stability notions for systems with inputs that are uniform with respect to initial time, such as uniform ISS and iISS, bound the state trajectory in relation to initial state, elapsed time and input. In the context of impulsive systems, the input can be interpreted as having both a continuous-time and an impulsive component. From (1b), one observes that the values of u at the instants $t \in \sigma$ may instantaneously affect the state trajectory. For this reason, input bounds suitable for the required stability properties have to account for the instantaneous values $u(t)$ at $t \in \sigma$. Given an input $w = (u, \sigma)$ and $\rho_1, \rho_2 \in \mathcal{K}_\infty$, we thus define

$$\|w\|_{(\rho_1,\rho_2)} := \int_0^\infty \rho_1(|u(s)|)ds + \sum_{t \in \sigma} \rho_2(|u(t)|). \tag{3}$$

The quantity defined in (3) can be loosely interpreted as a measure of the energy content of an input that has some impulsive behaviour at the time instants $t \in \sigma$.

We are interested in determining whether some stability property holds not just for a single impulse time sequence $\sigma \in \mathcal{I}$ but also for some family $\mathcal{S} \subset \mathcal{I}$. For example, if we know that jumps will not occur closer than $\Delta > 0$ units of time apart, the family $\mathcal{S}$ could be defined as all those sequences $\sigma = \{\tau_k\}_{k=1}^N$ where $\tau_{k+1} - \tau_k \geq \Delta$ for all $k \geq 1$. We thus consider the uniform stability notions given in Definition 2.1. To simplify notation, for every interval $J \subset [0, \infty)$ and $u \in \mathcal{U}$, we define u_J via $u_I(t) := u(t)$ if $t \in J$ and $u_J(t) := 0$ otherwise; for an input $w = (u, \sigma)$, we define $w_J := (u_J, \sigma)$.

Definition 2.1: Given $\mathcal{S} \subset \mathcal{I}$, we say that the impulsive system (1) is

a) 0-GUAS uniformly over (the family of impulse time sequences) $\mathcal{S}$ if there exists $\beta \in \mathcal{KL}$ such that
$$|x(t)| \leq \beta(|x(t_0)|, t - t_0) \quad \forall t \geq t_0, \tag{4}$$
for every $x \in \mathcal{T}(t_0, x_0, w_0)$ with $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ and $w_0 = (0, \sigma)$ with $\sigma \in \mathcal{S}$.

b) UBEBS uniformly over $\mathcal{S}$ if there exist $\alpha, \rho_1, \rho_2 \in \mathcal{K}_\infty$ and $c \geq 0$ such that
$$\alpha(|x(t)|) \leq |x(t_0)| + \|w_{(t_0,t]}\|_{(\rho_1,\rho_2)} + c \quad \forall t \geq t_0, \tag{5}$$
for every $x \in \mathcal{T}(t_0, x_0, w)$ with $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ and $w \in \mathcal{U} \times \mathcal{S}$. The pair (ρ_1, ρ_2) will be referred to as an UBEBS gain.

c) iISS uniformly over $\mathcal{S}$ if there exist $\beta \in \mathcal{KL}$ and $\alpha, \rho_1, \rho_2 \in \mathcal{K}_\infty$ such that
$$\alpha(|x(t)|) \leq \beta(|x(t_0)|, t - t_0) + \|w_{(t_0,t]}\|_{(\rho_1,\rho_2)} \tag{6}$$
for all $t \geq t_0$, for every $x \in \mathcal{T}(t_0, x_0, w)$ with $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ and $w \in \mathcal{U} \times \mathcal{S}$. The pair (ρ_1, ρ_2) will be referred to as an iISS gain.

If (1) is 0-GUAS uniformly over $\mathcal{S}$, then under $u \equiv 0$ the state converges asymptotically to the origin. In addition, this convergence is uniform over initial times and over impulse time sequences within the family $\mathcal{S}$. The uniform-over-$\mathcal{S}$ UBEBS property just imposes a bound on the state trajectory without necessarily guaranteeing convergence. The bound is uniform over initial times and over all $\sigma \in \mathcal{S}$, and depends on the initial state norm and the input energy. The uniform-over-$\mathcal{S}$ iISS property imposes a bound that is also uniform over initial times and over all $\sigma \in \mathcal{S}$. This bound is formed by a term similar to the 0-GUAS property and another term equal to the input energy.

This paper aims at yielding insight into this problem:

> Let $\mathcal{S} \subset \mathcal{I}$. Under what conditions is the uniform-over-$\mathcal{S}$ iISS property equivalent to the combination of 0-GUAS and UBEBS, both uniformly over $\mathcal{S}$?

This problem was answered for time-invariant non-impulsive systems in [4], and for time-varying and switched (non-impulsive) systems in [9].

III. Main Results

A. Previous assumptions

First, we note that if jumps do not occur ($\sigma = \emptyset$), then (1) becomes the type of system considered in [9] and hence f in (1a) has to satisfy the conditions required in Assumption 1 of [9]. We thus require the following definiton.

Definition 3.1: A function $h : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is said to belong to class $\mathcal{A}$, written $h \in \mathcal{A}$, if the following items hold:

i) there exist $\nu_h \in \mathcal{K}$ and a nondecreasing function $N_h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ such that $|h(t,\xi,\mu)| \leq N_h(|\xi|)(1+\nu_h(|\mu|))$ for all $t \geq 0$, all $\xi \in \mathbb{R}^n$ and all $\mu \in \mathbb{R}^m$;

ii) for every $r > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t \geq 0$, $|h(t,\xi,\mu) - h(t,\xi,0)| < \varepsilon$ if $|\xi| \leq r$ and $|\mu| \leq \delta$.

It is said to belong to class $\mathcal{AL}$, written $h \in \mathcal{AL}$, if $h \in \mathcal{A}$ and, in addition,

iii) $h(t,\xi,0)$ is locally Lipschitz in ξ, uniformly in t, i.e. for every $\xi \in \mathbb{R}^n$ there are an open ball B containing ξ and a constant $L \geq 0$ so that for every $\xi_1, \xi_2 \in B$ and $t \geq 0$ it happens that $|h(t,\xi_1,0) - h(t,\xi_2,0)| \leq L|\xi_1 - \xi_2|$.

Note that a function h belongs to class $\mathcal{AL}$ if and only if it verifies Assumption 1 of [9]. We thus will require that f in (1a) satisfy $f \in \mathcal{AL}$.

B. Intermediate results

To see what other conditions may be needed, we will try to follow, mutatis mutandis, the steps in the proof of [9, Theorem 1] for the characterization of iISS for non-impulsive time-varying and switched systems. This proof requires two intermediate results, namely Lemmas 3 and 4 in [9]. Lemma 3 in [9] gives a very crude estimate on how large the state of a 0-GUAS system can become for an arbitrary input, depending on time and input energy, when it is known that the state remains bounded. The estimate has the property that the bound on the state norm depends only on the initial state $x(t_0)$ and the elapsed time $t - t_0$, but is independent of (i.e. uniform over) the initial time t_0. The proof of [9, Lemma 3] requires the integral expression for the solution to $\dot{x} = f(t,x,u)$, given by $x(t) = x(t_0) + \int_{t_0}^t f(s,x(s),u(s))ds$. As compared with the latter, the integral expression for the solution of (1) has an extra term due to its impulsive part:

$$x(t) = x(t_0) + \int_{t_0}^t f(s,x(s),u(s))ds + \sum_{\tau \in \sigma \cap (t_0,t]} g(\tau, x(\tau^-), u(\tau)). \quad (7)$$

The proof of Lemma 3 of [9] also requires Gronwall inequality, which provides an explicit inequality for continuous functions that satisfy an implicit integral inequality. The next lemma provides a generalization of this inequality suitable for piecewise-continuous functions, i.e. for continuous functions with isolated jumps.

Lemma 3.1 (Generalized Gronwall Inequality): Let $0 \leq t_0 < T \leq \infty$ and let $y : [t_0,T) \to \mathbb{R}$ be a right-continuous function having a finite left-limit at every discontinuity instant. Suppose that the points of discontinuity of y can be arranged into a sequence $\sigma \in \mathcal{I}$. Let $p \in \mathbb{R}$ and $q_1, q_2 \geq 0$. If y satisfies

$$y(t) \leq p + q_1 \int_{t_0}^t y(s)ds + q_2 \sum_{s \in \sigma \cap (t_0,t]} y(s^-) \quad (8)$$

for all $t \in [t_0,T)$, then in the same time interval y also satisfies

$$y(t) \leq p(1+q_2)^{n^\sigma_{(t_0,t]}} \cdot e^{q_1(t-t_0)}. \quad (9)$$

The proof of Lemma 3.1 follows by direct application of Proposition 1 of [12] (see also Theorem 1.5.1 in [10]). Note that if y is continuous and hence σ in Lemma 3.1 satisfies $\sigma = \emptyset$, then Lemma 3.1 reduces to the classic Gronwall inequality.

We are now ready to provide a generalization of Lemma 3 of [9] in the current setting. The proof is given in the Appendix.

Lemma 3.2 (Generalization of Lemma 3 in [9]): Let $\mathcal{S} \subset \mathcal{I}$, let the impulsive system (1) be 0-GUAS uniformly over $\mathcal{S}$ and let $\beta \in \mathcal{KL}$ characterize the 0-GUAS property. Suppose that $f, g \in \mathcal{AL}$ and let ν_f and ν_g be, respectively, the functions corresponding to f and g as per item i) of Definition 3.1. Let $\chi_f, \chi_g \in \mathcal{K}_\infty$ satisfy $\chi_f \geq \nu_f$ and $\chi_g \geq \nu_g$. Then, for every $r > 0$ and every $\eta > 0$, there exists $L = L(r)$ and $\kappa = \kappa(r,\eta)$ such that if $x \in \mathcal{T}(t_0,x_0,w)$ with $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$, $w = (u,\sigma) \in \mathcal{U} \times \mathcal{S}$ satisfies $|x(t)| \leq r$ for all $t \geq t_0$, then also

$$|x(t)| \leq \beta(|x_0|, t-t_0) + \left[(t-t_0+n^\sigma_{(t_0,t]})\eta + \kappa \|w_{(t_0,t]}\|_{(\chi_f,\chi_g)}\right](1+L)^{n^\sigma_{(t_0,t]}} \cdot e^{L(t-t_0)}. \quad (10)$$

As with [9, Lemma 3], the inequality (10) is only useful when its right-hand side is less than r, since $|x(t)| \leq r$ for all $t \geq t_0$ is already assumed. If $\sigma = \emptyset$ (no impulses), and hence $n^\sigma_{(t_0,t]} = 0$, then (10) reduces to the corresponding bound in Lemma 3 of [9].

We now reach the main difference between the non-impulsive and impulsive cases, namely that the bound on the state given by Lemma 3.2 does not only depend on the initial state and elapsed time. This happens because of the factor $n^\sigma_{(t_0,t]}$ that counts the number of jumps in the interval $(t_0,t]$. Fixing the elapsed time to T, so that $t = t_0 + T$, the quantity $n^\sigma_{(t_0,t_0+T]}$ need not be uniformly bounded over all values of t_0, as the following example shows.

Example 3.1: Consider the sequence $\sigma = \{\tau_k\}_{k=1}^\infty$ with $t_1 = 1$ and $\tau_{k+1} = \tau_k + 1/(k+1)$. Note that σ is a strictly increasing sequence and $\lim_{k\to\infty} \tau_k = \sum_{k=1}^\infty (1/k) = \infty$. Then σ has no finite limit points and hence $\sigma \in \mathcal{I}$. However, $\lim_{k\to\infty} \tau_{k+1} - \tau_k = \lim_{k\to\infty} 1/(k+1) = 0$, and hence consecutive elements of σ occur closer together as time increases. Then, if we consider the interval $(t_0, t_0+1]$ and the number of elements of σ that fall within the latter interval, namely $n^\sigma_{(t_0,t_0+1]}$, it follows that $\lim_{t_0\to\infty} n^\sigma_{(t_0,t_0+1]} = \infty$. $\circ$

We thus will require the following definition.

Definition 3.2: The set $\mathcal{S} \subset \mathcal{I}$ is said to be uniformly incrementally bounded (UIB) if there exists a nondecreasing function $\phi : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ so that $n^{\sigma}_{(t_0,t]} \leq \phi(t - t_0)$ for every $\sigma \in \mathcal{S}$ and all $t > t_0 \geq 0$. ◦

The set $\mathcal{S}[\tau_D]$ of dwell-time sequences with dwell-time $\tau_D > 0$ or more generally the set $\mathcal{S}[N_0, \tau_D]$ of average dwell-time sequences with chatter bound $N_0 \in \mathbb{N}$ and average dwell-time $\tau_D > 0$ ($\sigma \in \mathcal{S}[N_0, \tau_D]$ if $n^{\sigma}_{(t_0,t]} \leq N_0 + \tau_D(t - t_0)$ for all $0 \leq t_0 < t$), are examples of UIB sets.

The other intermediate result, namely Lemma 4 of [9], shows that if a system is 0-GUAS, then UBEBS could be equivalently defined setting $c = 0$ in (5). By imposing the condition that $\mathcal{S} \subset \mathcal{I}$ be UIB, we are able to extend this result to the current setting. The proof is given in the Appendix.

Lemma 3.3: Consider the impulsive system (1) and let $\mathcal{S} \subset \mathcal{I}$ be UIB. Suppose that $f, g \in \mathcal{AL}$. If (1) is 0-GUAS and UBEBS, both uniformly over $\mathcal{S}$, then there exist $\tilde{\alpha}, \tilde{\rho}_1, \tilde{\rho}_2 \in \mathcal{K}_\infty$ for which the estimate (11) holds for every $x \in \mathcal{T}(t_0, x_0, w)$ with $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ and $w \in \mathcal{U} \times \mathcal{S}$.

$$\tilde{\alpha}(|x(t)|) \leq |x(t_0)| + \|w_{(t_0,t]}\|_{(\tilde{\rho}_1,\tilde{\rho}_2)} \quad \forall t \geq t_0. \qquad (11)$$

C. Characterizations of iISS

We now have almost all the ingredients required for answering the stated problem. The only additional step is an ϵ-δ characterization of the uniform-over-$\mathcal{S}$ iISS property, given as Theorem 3.1. The proof is given in the Appendix.

Theorem 3.1: Let $\rho_1, \rho_2 \in \mathcal{K}_\infty$ and $\mathcal{S} \subset \mathcal{I}$. Consider the notation $\|w\| = \|w\|_{(\rho_1,\rho_2)}$ and for $r \geq 0$, $B^{\mathcal{S}}_r := \{w \in \mathcal{U} \times \mathcal{S} : \|w\| \leq r\}$. Then, system (1) is iISS uniformly over $\mathcal{S}$ with iISS gain (ρ_1, ρ_2) if and only if it is forward complete for every $\sigma \in \mathcal{S}$ and the following conditions hold:

i) For every $T \geq 0$, $r \geq 0$, $s \geq 0$, there exists $C > 0$ such that every $x \in \mathcal{T}(t_0, x_0, w)$ with $t_0 \geq 0$, $x_0 \in B^n_r$ and $w \in B^{\mathcal{S}}_s$ satisfies $|x(t)| \leq C$ for all $t \in [t_0, t_0 + T]$.

ii) For each $\epsilon > 0$, there exists $\delta > 0$ such that every $x \in \mathcal{T}(t_0, x_0, w)$ with $t_0 \geq 0$, $x_0 \in B^n_\delta$ and $w \in B^{\mathcal{S}}_\delta$ satisfies $|x(t)| \leq \epsilon$ for all $t \geq t_0$.

iii) There exists $\tilde{\alpha} \in \mathcal{K}_\infty$ such that for every $r, \epsilon > 0$ there exists $T > 0$ so that for every $x \in \mathcal{T}(t_0, x_0, w)$ with $t_0 \geq 0$, $x_0 \in B^n_r$ and $w \in \mathcal{U} \times \mathcal{S}$, then

$$\tilde{\alpha}(|x(t)|) \leq \epsilon + \|w\|, \quad \forall t \geq t_0 + T.$$

We may finally provide an answer to the problem addressed. This is given in the following Theorem.

Theorem 3.2: Consider the impulsive system (1) and suppose that $f, g \in \mathcal{AL}$. Let $\mathcal{S} \subset \mathcal{I}$ be a UIB set of impulse time sequences. Then, (1) is iISS uniformly over $\mathcal{S}$ if and only if it is 0-GUAS and UBEBS, both uniformly over $\mathcal{S}$.

Proof: ($\Rightarrow$) Considering $w = (u, \gamma)$ with $u = 0$, the estimate (6) reduces to $\alpha(|x(t)|) \leq \beta(|x(t_0)|, t - t_0)$ and hence $|x(t)| \leq \alpha^{-1}(\beta(|x(t_0)|, t - t_0))$. The function $\tilde{\beta} := \alpha^{-1} \circ \beta$ satisfies $\tilde{\beta} \in \mathcal{KL}$, and hence (4) follows with β replaced by $\tilde{\beta}$. Therefore, clearly iISS implies 0-GUAS, both uniformly over $\mathcal{S}$. Consider $\beta \in \mathcal{KL}$ from (6), define $\beta_0 \in \mathcal{K}_\infty$ via $\beta_0(r) = \beta(r, 0)$, and note that $\beta_0(r) \geq r$ for all $r \geq 0$. Define $\psi \in \mathcal{K}_\infty$ via $\psi(r) = \beta_0^{-1}(r/2) \in \mathcal{K}_\infty$, and note that $\psi(r) \leq r/2$ for all $r \geq 0$. From (6), we obtain

$$\begin{aligned}\psi \circ \alpha(|x(t)|) &\leq \psi\left(\beta_0(|x(t_0)|) + \|w_{(t_0,t]}\|_{(\rho_1,\rho_2)}\right) \\ &\leq \psi\left(2\beta_0(|x(t_0)|)\right) + \psi(2\|w_{(t_0,t]}\|_{(\rho_1,\rho_2)}) \\ &\leq |x(t_0)| + \|w_{(t_0,t]}\|_{(\rho_1,\rho_2)},\end{aligned}$$

and hence (5) follows with α replaced by $\tilde{\alpha} := \psi \circ \alpha \in \mathcal{K}_\infty$. We have thus shown that iISS implies UBEBS, both uniformly over $\mathcal{S}$.

($\Leftarrow$) Let $\tilde{\alpha}, \tilde{\rho}_1, \tilde{\rho}_2 \in \mathcal{K}_\infty$ be given by Lemma 3.3, so that (11) is satisfied. We will prove that (1) is iISS uniformly over $\mathcal{S}$ with iISS gain $(\tilde{\rho}_1, \tilde{\rho}_2)$ by establishing each of the items of Theorem 3.1.

i) Let $T \geq 0$, $r \geq 0$ and $s \geq 0$. Let $x \in \mathcal{T}(t_0, x_0, w)$ with $t_0 \geq 0$, $x_0 \in B^n_r$, $w \in B^{\mathcal{S}}_s$. From (11), it follows that $\tilde{\alpha}(|x(t)|) \leq r + s$, and hence $|x(t)| \leq \tilde{\alpha}^{-1}(r + s) + 1 =: C$ for all $t \geq t_0$. This establishes item i) of Theorem 3.1.

ii) Let $\epsilon > 0$. Let $\delta = \tilde{\alpha}(\epsilon)/2$. Then, if $x \in \mathcal{T}(t_0, x_0, w)$ with $t_0 \geq 0$, $x_0 \in B^n_\delta$ and $w \in B^{\mathcal{S}}_\delta$, it follows from (11) that $|x(t)| \leq \tilde{\alpha}^{-1}(2\delta) = \epsilon$ for all $t \geq t_0$. This establishes item ii) of Theorem 3.1.

iii) Let $\alpha = \tilde{\alpha}/2 \in \mathcal{K}_\infty$. Let $r, \epsilon > 0$ and let $x \in \mathcal{T}(t_0, x_0, w)$ with $t_0 \geq 0$, $x_0 \in B^n_r$ and $w \in \mathcal{U} \times \mathcal{S}$. We distinguish two cases:

(a) $\|w\| \geq r$,

(b) $\|w\| < r$.

In case (a), from (11) we have $\tilde{\alpha}(|x(t)|) \leq r + \|w_{(t_0,t]}\| \leq r + \|w\| \leq 2\|w\|$, hence $\alpha(|x(t)|) \leq \|w\| \leq \epsilon + \|w\|$ for all $t \geq t_0$.

Next, consider case (b). From (11), we have $\tilde{\alpha}(|x(t)|) \leq r + \|w\| < 2r =: \tilde{r}$ for all $t \geq t_0$. Let $\beta \in \mathcal{KL}$ characterize uniform-over-$\mathcal{S}$ 0-GUAS property, so that (4) is satisfied under zero input, and let $L = L(\tilde{r}) > 0$ be given by Lemma 3.2. Let $\tilde{\epsilon} = \epsilon$ and $\tilde{T} > 0$ satisfy $\beta(\tilde{r}, \tilde{T}) < \tilde{\epsilon}/2$. Let ϕ correspond to the UIB property of $\mathcal{S}$, and define $\psi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ via $\psi(s) = s + \phi(s)$. Define $\eta = \frac{\tilde{\epsilon}}{4\psi(\tilde{T})(1+L)^{\psi(\tilde{T})}e^{L\psi(\tilde{T})}}$. Let $\kappa = \kappa(\tilde{r}, \eta) > 0$ be given by Lemma 3.2. Let $\delta = \frac{\tilde{\epsilon}}{4\kappa(1+L)^{\psi(\tilde{T})}e^{L\psi(\tilde{T})}}$. Define $N := \lceil \frac{r}{\delta} \rceil$ and $T := N\tilde{T}$, where $\lceil s \rceil$ denotes the least integer not less than $s \in \mathbb{R}$.

For $i = 0$ to N, let $s_i = t_0 + i\tilde{T}$. Consider the intervals $I_i = [s_{i-1}, s_i]$, with $i = 1, \ldots, N$. We claim that there exists $j \leq N - 1$ for which $\|w_{(s_j,s_{j+1}]}\| \leq \delta$. For a contradiction, suppose that $\|w_{(s_j,s_{j+1}]}\| > \delta$ for all $0 \leq j \leq N - 1$. Then, $\|w\| \geq \|w_{(s_0,s_N]}\| = \sum_{j=0}^{N-1} \|w_{(s_j,s_{j+1}]}\| > N\delta \geq r$, contradicting case (b). Therefore, let $0 \leq j \leq N - 1$ be such that $\|w_{(s_j,s_{j+1}]}\| \leq \delta$.

Since $x \in \mathcal{T}(s_j, x(s_j), w)$ and $|x(t)| \leq \tilde{r}$ for all $t \geq s_j$, from Lemma 3.2 it follows that

$$\begin{aligned}&|x(s_j + \tilde{T})| \leq \beta(|x(s_j)|, \tilde{T}) + \\ &\left([\tilde{T} + n^{\sigma}_{(s_j,s_{j+1}]}]\eta + \kappa\|w_{(s_j,s_{j+1}]}\|\right)(1 + L)^{n^{\sigma}_{(s_j,s_{j+1}]}} e^{L\tilde{T}} \\ &\qquad \leq \beta(\tilde{r}, \tilde{T}) + (\psi(\tilde{T})\eta + \kappa\delta)(1 + L)^{\psi(\tilde{T})} e^{L\psi(\tilde{T})} \leq \tilde{\epsilon}.\end{aligned}$$

Therefore, using (11) with t_0 replaced by $s_j+\tilde{T}$, we reach

$$\tilde{\alpha}(|x(t)|) \le |x(s_j+\tilde{T})| + \|w_{(s_j+\tilde{T},t]}\| \le \tilde{\epsilon} + \|w\|$$

for all $t \ge t_0+T$ because $t_0+T \ge s_j+\tilde{T}$. Since $\alpha = \tilde{\alpha}/2 \le \tilde{\alpha}$, it follows that item iii) of Theorem 3.1 also is satisfied. ■

IV. Conclusions

We have addressed the characterization of the integral input-to-state stability property for time-varying impulsive systems in terms of global uniform asymptotic stability under zero input and a uniformly bounded-energy input bounded state property. We have shown that by imposing the condition that the number of jumps over an interval be bounded in relation to the interval length but irrespective of the interval's initial time, this characterization carries over to the type of time-varying impulsive systems considered. Future work may be aimed at extending this characterization to hybrid systems where jumps could occur infinitely often.

Appendix

A. Proof of Lemma 3.2

The proof requires the following Claim, whose proof follows from Appendix B of [9] by replacing $f(t,\xi,\mu,i)$ with $h(t,\xi,\mu)$ and γ with ν_h.

Claim 1: Let $h \in \mathcal{A}$ (as per Definition 3.1). Then, for every $r^* > 0$ and $\eta > 0$ there exists $\kappa = \kappa(r^*,\eta) > 0$ such that for all $t \ge 0$, $\xi \in B^n_{r^*}$ and $\mu \in \mathbb{R}^m$,

$$|h(t,\xi,\mu) - h(t,\xi,0)| \le \eta + \kappa\nu_h(|\mu|).$$

Fix $r > 0$ and $\eta > 0$, and define $r^* := \beta(r,0) \ge r$. Let $L_f = L_f(r) > 0$ and $L_g = L_g(r) > 0$ be Lipschitz constants for $f(t,\cdot,0)$ and $g(t,\cdot,0)$, respectively, on the compact set $B^n_{r^*}$ and valid for every $t \ge 0$. Let $L = \max\{L_f, L_g\}$. Let κ_f and κ_g be the quantities given by Claim 1 in correspondence with r^* and η, for f and g, respectively. Let $\kappa = \max\{\kappa_f,\kappa_g\}$. Let $x \in \mathcal{T}(t_0,x_0,w)$ with $t_0 \ge 0$, $x_0 \in \mathbb{R}^n$, $w = (u,\sigma) \in \mathcal{U}\times\mathcal{S}$ satisfy $|x(t)| \le r$ for all $t \ge t_0$. Let $y \in \mathcal{T}(t_0,x_0,w_0)$, with $w_0 = (0,\sigma)$. Then, $x(t), y(t) \in B^n_{r^*}$ for all $t \ge t_0$. Let $t \ge t_0$. For all $t_0 \le \tau \le t$, we have, using (7),

$$|x(\tau)-y(\tau)| \le \int_{t_0}^{\tau} \Big| f\big(s,x(s),u(s)\big) - f\big(s,y(s),0\big)\Big| ds + \sum_{s\in\sigma\cap(t_0,\tau]} \Big| g\big(s,x(s^-),u(s)\big) - g\big(s,y(s^-),0\big)\Big|$$

Adding and subtracting $f(s,x(s),0)$ and $g(s,x(s^-),0)$ within the respective norm signs, and employing the bound in f and g given by Claim 1, it follows that

$$|f(s,x(s),u(s)) - f(s,y(s),0)| \le \eta + \kappa_f\nu_f(|u(s)|) + L_f|x(s)-y(s)|,$$
$$|g(s,x(s^-),u(s)) - g(s,y(s^-),0)| \le \eta + \kappa_g\nu_g(|u(s)|) + L_g|x(s^-)-y(s^-)|.$$

Defining $\delta(t) = |x(t)-y(t)|$, and recalling the definition of L and κ, then for all $t_0 \le \tau \le t$,

$$\delta(\tau) \le \int_{t_0}^{t} [\eta + \kappa\chi_f(|u(s)|)] ds + \sum_{s\in\sigma\cap(t_0,t]} [\eta+\kappa\chi_g(|u(s)|)] + \int_{t_0}^{\tau} L\delta(s)ds + \sum_{s\in\sigma\cap(t_0,\tau]} L\delta(s^-)$$

The result then follows from application of Lemma 3.1, the facts that $\int_{t_0}^t \eta ds = (t-t_0)\eta$ and $\sum_{s\in\sigma\cap(t_0,t]}\eta = n^\sigma_{(t_0,t]}\eta$, and $|x(t)| \le |y(t)| + \delta(t) \le \beta(|x_0|, t-t_0) + \delta(t)$.

B. Proof of Lemma 3.3

Let α, ρ_1, ρ_2 and c be as in the estimate (5). Let $\tilde{\rho}_1 := \max\{\rho_1,\nu_f\}$ and $\tilde{\rho}_2 := \max\{\rho_2,\nu_g\}$. For $r \ge 0$ define

$$\bar{\alpha}(r) := \sup_{x\in\mathcal{T}(t_0,x_0,w),\ t\ge t_0\ge 0,\ |x_0|\le r,\ w\in\mathcal{U}\times\mathcal{S},\ \|w\|\le r} |x(t)|$$

where $\|w\| := \|w\|_{(\tilde{\rho}_1,\tilde{\rho}_2)}$. From this definition, it follows that $\bar{\alpha}$ is nondecreasing and from (5) that it is finite for all $r \ge 0$. Next, we show that $\lim_{r\to 0^+}\bar{\alpha}(r) = 0$. Let $\beta \in \mathcal{KL}$ be the function which characterizes the uniform-over-$\mathcal{S}$ 0-GUAS property of (1), let ϕ correspond to the UIB property of $\mathcal{S}$, and define $\psi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ via $\psi(s) = s + \phi(s)$. Let $r^* = \alpha^{-1}(2+c)$ and $L = L(r^*) > 0$ be given by Lemma 3.2. Let $\varepsilon > 0$ be arbitrary. Pick $0 < \delta_1 < 1$ such that $\delta_1 \le \beta(\delta_1,0) < \varepsilon/2$, and $T > 0$ such that $\beta(\delta_1,T) < \delta_1/2$. Define $\eta = \frac{\delta_1}{4\psi(T)(1+L)^{\psi(T)}e^{L\psi(T)}}$ and let $\kappa = \kappa(r^*,\eta) > 0$ be given by Lemma 3.2. Last, pick $0 < \delta_2 < 1$ such that $\delta_2 < \frac{\delta_1}{4\kappa(1+L)^{\psi(T)}e^{L\psi(T)}}$. Then, for every $x \in \mathcal{T}(t_0,x_0,w)$, with $t_0 \ge 0$, $|x_0| \le \delta_1$, $w \in \mathcal{U}\times\mathcal{S}$ and $\|w\| \le \delta_2$, we claim that $|x(t)| < \varepsilon$ for all $t \ge t_0$. First, note that under the given bounds for x_0 and w, from (5) it follows that $\alpha(|x(t)|) \le \delta_1 + \delta_2 + c \le 2 + c$, and hence $|x(t)| \le r^*$ for all $t \ge t_0 \ge 0$. Then, application of Lemma 3.2 with $\chi_f = \tilde{\rho}_1$ and $\chi_g = \tilde{\rho}_2$ gives the estimate (10), whence for all $t \in [t_0, t_0+T]$,

$$|x(t)| \le \beta(|x_0|,0) + [\psi(T)\eta + \kappa\|w_{(t_0,t]}\|](1+L)^{\psi(T)}e^{L\psi(T)}$$
$$\le \beta(\delta_1,0) + [\psi(T)\eta + \kappa\delta_2](1+L)^{\psi(T)}e^{L\psi(T)} < \varepsilon,$$

where we have used the facts that $0 \le t - t_0 \le T \le \psi(T)$, $n^\sigma_{(t_0,t]} \le \phi(t-t_0) \le \phi(T) \le \psi(T)$. Also from (10) we have that $|x(t_0+T)| \le \beta(\delta_1,T) + [\psi(T)\eta + \kappa\delta_2](1+L)^{\psi(T)}e^{L\psi(T)} < \delta_1$. Since $x \in \mathcal{T}(s_1, x(s_1), w)$, with $s_1 = t_0+T$, and $|x(s_1)| < \delta_1$, then $|x(t)| < \varepsilon$ for all $t \in [s_1, s_1+T]$ and $|x(s_1+T)| < \delta_1$. Therefore, by means of an inductive argument we can prove that $|x(t)| < \varepsilon$ for all $t \in [s_n, s_n+T]$, where $s_n = t_0 + nT$, and that $|x(s_n+T)| < \delta_1$. In consequence, $|x(t)| < \varepsilon$ for all $t \ge t_0$ as we claim. Thus, if $\delta = \min\{\delta_1,\delta_2\}$, for all $x \in \mathcal{T}(t_0,x_0,w)$, with $t_0 \ge 0$, $|x_0| \le \delta$, $w \in \mathcal{U}\times\mathcal{S}$ with $\|w\| \le \delta$, we have $|x(t)| \le \varepsilon$ for all $t \ge t_0$. Therefore, $\bar{\alpha}(r) \le \bar{\alpha}(\delta) < \varepsilon$ for all $0 < r < \delta$ and $\lim_{r\to 0^+}\bar{\alpha}(r) = 0$.

Since $\bar{\alpha}$ is nondecreasing and $\lim_{r\to 0^+}\bar{\alpha}(r) = 0$ there exists $\hat{\alpha} \in \mathcal{K}_\infty$ such that $\hat{\alpha}(r) \ge \bar{\alpha}(r)$ for all $r \ge 0$. Let $x \in \mathcal{T}(t_0,x_0,w)$ with $t_0 \ge 0$, $x_0 \in \mathbb{R}^n$ and $w \in \mathcal{U}\times\mathcal{S}$. Let

$t \geq t_0$.From well-known results on differential equations, there exists $x^* \in \mathcal{T}(t_0, x_0, w_{(t_0,t]})$ such that $x^*(\tau) = x(\tau)$ for all $\tau \in [t_0, t]$. By using the definition of $\bar{\alpha}$ and the fact that $\hat{\alpha}(r) \geq \bar{\alpha}(r)$, we then have $|x(t)| = |x^*(t)| \leq \hat{\alpha}(|x_0|) + \hat{\alpha}(\|w_{(t_0,t]}\|)$. Define $\tilde{\alpha} \in \mathcal{K}_\infty$ via $\tilde{\alpha}(s) = \hat{\alpha}^{-1}(s)/2$. Applying $\tilde{\alpha}$ to both sides of the preceding inequality and using the fact that $\tilde{\alpha}(a+b) \leq \tilde{\alpha}(2a) + \tilde{\alpha}(2b)$, we reach $\tilde{\alpha}(|x(t)|) \leq |x_0| + \|w_{(t_0,t]}\|$, which establishes the result.

C. Proof of Theorem 3.1

Necessity is straightforward, so we just establish sufficiency. Item i) implies that (1) is forward complete for every $\sigma \in \mathcal{S}$. To see this, suppose that there exist $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ and $w \in \mathcal{U} \times \mathcal{S}$ so that $x \in \mathcal{T}(t_0, x_0, w)$ is not forward complete. Let $T_m < \infty$ be the maximum existence time of x. Then, $\sigma \cap [t_0, T_m)$ is finite and x is obtained by piecing together a finite number of Carathéodory solutions over bounded intervals. By standard properties of differential equations, then $\lim_{t \nearrow T_m} |x(t)| = \infty$. By causality, x does not change if w is replaced by $\bar{w} := w_{[t_0,T_m]} = (u_{[t_0,T_m]}, \sigma_{[t_0,T_m]})$, with $u_{[t_0,T_m]}$ coinciding with u in the interval $[t_0, T_m]$ and being 0 elsewhere, and $\sigma_{[t_0,T_m]} = \sigma \cap [t_0, T_m]$. Since $u \in \mathcal{U}$ implies that u is locally bounded, it follows that $s := \|\bar{w}\| < \infty$. Let $r = |x(t_0)|$ and take C in correspondence with r, s and $T = T_m$ from item i). Then, $|x(t)| \leq C$ for all $t \in [t_0, t_0 + T_m]$, which contradicts the fact that $\lim_{t \nearrow T_m} |x(t)| = \infty$.

Let $\tilde{\alpha} \in \mathcal{K}_\infty$ and $T > 0$ be given by item iii), the latter in correspondence with $r > 0$ and $\epsilon = 1$. Let C be given by item i) in correspondence with $s = r$ and T. From items i) and iii), we then have, whenever $t_0 \geq 0$, $x_0 \in B_r^n$ and $w \in B_r^{\mathcal{S}}$,

$$|x(t)| \leq C, \quad \forall t \in [t_0, t_0 + T],$$
$$\tilde{\alpha}(|x(t)|) \leq 1 + \|w\|, \quad \forall t \geq t_0 + T.$$

It follows that $\tilde{\alpha}(|x(t)|) \leq \tilde{\alpha}(C) + 1 + \|w\|$ for all $t \geq t_0$.

Let $\phi(r) := \inf\{\tilde{C} \geq 0 : \tilde{\alpha}(|x(t)|) \leq \tilde{C},\ \forall x \in \mathcal{T}(t_0, x_0, w), \forall t \geq t_0 \geq 0, \forall x_0 \in B_r^n, \forall w \in B_r^{\mathcal{S}}\}$. By the previous analysis, then $\phi(r) \leq \tilde{\alpha}(C) + 1 + r < \infty$ for all $r \geq 0$. Also, ϕ is nondecreasing and $\tilde{\alpha}(|x(t)|) \leq \phi(|x(t_0)|) + \phi(\|w\|)$ for all $t \geq t_0$ whenever $x \in \mathcal{T}(t_0, x_0, w)$ with $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ and $w \in \mathcal{U} \times \mathcal{S}$. From item ii), it follows that $\lim_{r \searrow 0} \phi(r) = 0$. There thus exists $\eta \in \mathcal{K}_\infty$ such that $\phi \leq \eta$ and then

$$\tilde{\alpha}(|x(t)|) \leq \eta(|x(t_0)|) + \eta(\|w\|) \quad \text{for all } t \geq t_0, \tag{12}$$

whenever $x \in \mathcal{T}(t_0, x_0, w)$ with $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ and $w \in \mathcal{U} \times \mathcal{S}$. Let $\psi, \alpha \in \mathcal{K}_\infty$ be defined via $\psi(s) = \eta^{-1}(s/2)$ and $\alpha = \min\{\tilde{\alpha}, \psi \circ \tilde{\alpha}\}$. Then, applying ψ to (12) and using the inequality $\psi(a+b) \leq \psi(2a) + \psi(2b)$, it follows that

$$\alpha(|x(t)|) \leq |x(t_0)| + \|w\| \quad \text{for all } t \geq t_0, \tag{13}$$

whenever $x \in \mathcal{T}(t_0, x_0, w)$ with $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ and $w \in \mathcal{U} \times \mathcal{S}$. Define

$$T_{r,\epsilon} := \inf\big\{\tau \geq 0 : \alpha(|x(t)|) \leq \epsilon + \|w\|, \forall t \geq t_0 + \tau,$$
$$\forall t_0 \geq 0, \forall x \in \mathcal{T}(t_0, x_0, w), \forall x_0 \in B_r^n, \forall w \in \mathcal{U} \times \mathcal{S}\big\}.$$

By item iii) and since $\alpha \leq \tilde{\alpha}$, then $T_{r,\epsilon} < \infty$ for every $r, \epsilon > 0$. Moreover, $T_{r,\epsilon}$ is nondecreasing in r for fixed $\epsilon > 0$ and nonincreasing in ϵ for fixed $r > 0$. By (13), then $T_{r,\epsilon} \to 0$ as $\epsilon \to \infty$ for fixed $r > 0$.

Fact 1: $T_{r,\epsilon}$ can be strictly upper bounded by $\bar{T}_{r,\epsilon}$ with the following properties:

a) For each fixed $r > 0$, $\bar{T}_{r,\cdot} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is continuous, strictly decreasing, and onto, so that $\lim_{\epsilon \searrow 0} \bar{T}_{r,\epsilon} = \infty$ and $\lim_{\epsilon \to \infty} \bar{T}_{r,\epsilon} = 0$.
b) For each fixed $\epsilon > 0$, $\bar{T}_{\cdot,\epsilon}$ is strictly increasing and $\lim_{r \to \infty} \bar{T}_{r,\epsilon} = \infty$.

Let ψ_r denote the inverse function of $\bar{T}_{r,\epsilon}$ considered as a function of ϵ for fixed $r > 0$. For every $r > 0$, then ψ_r is continuous on $\mathbb{R}_{>0}$ and $\lim_{s \searrow 0} \psi_r(s) = \infty$. By definition of $T_{r,\epsilon}$ and since $\bar{T}_{r,\epsilon} > T_{r,\epsilon}$, we have that

$$t_0 \geq 0, x_0 \in B_r^n, w \in \mathcal{U} \times \mathcal{S}, x \in \mathcal{T}(t_0, x_0, w), t \geq t_0 + \bar{T}_{r,\epsilon}$$
$$\Rightarrow \quad \alpha(|x(t)|) \leq \epsilon + \|w\| \tag{14}$$

Note that $t = t_0 + \bar{T}_{r,\epsilon}$ is equivalent to $\epsilon = \psi_r(t - t_0)$. Hence, from the implication (14) at $t = t_0 + \bar{T}_{r,\epsilon}$, it follows that

$$t > t_0 \geq 0, x_0 \in B_r^n, w \in \mathcal{U} \times \mathcal{S}, x \in \mathcal{T}(t_0, x_0, w)$$
$$\Rightarrow \quad \alpha(|x(t)|) \leq \psi_r(t - t_0) + \|w\| \tag{15}$$

The proof concludes following exactly the same steps as for the proof of Lemma 2.7 in [13].

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