Brief paper

Nonrobustness of asymptotic stability of impulsive systems with inputs☆

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Keywords:
Impulsive systems
Nonlinear systems
Time-varying systems
Input-to-state stability

A B S T R A C T

Suitable continuity and boundedness assumptions on the function f defining the dynamics of a time-varying nonimpulsive system with inputs are known to make the system inherit stability properties from the zero-input system. Whether this type of robustness holds or not for impulsive systems was still an open question. By means of suitable (counter)examples, we show that such stability robustness with respect to the inclusion of inputs cannot hold in general, not even for impulsive systems with time-invariant flow and jump maps. In particular, we show that zero-input global uniform asymptotic stability (0-GUAS) does not imply converging input converging state (CICS), and that 0-GUAS and uniform bounded-energy input bounded state (UBEBS) do not imply integral input-to-state stability (iISS). We also comment on available existing results that, however, show that suitable constraints on the allowed impulse–time sequences indeed make some of these robustness properties possible.

1. Introduction

Impulsive systems are dynamic systems whose state evolves continuously most of the time but may exhibit jumps (discontinuities) at isolated time instants (Lakshmikantham et al., 1989). The set of time instants when jumps occur is part of the impulsive system definition. We consider impulsive systems where the continuous dynamics is governed by a differential equation, characterized by the flow map, and where the state value immediately after a jump is given by a static equation, namely the jump map.

The stability properties of impulsive systems with or without inputs depend on the interplay between the continuous and the impulsive behaviors (Hespanha et al., 2008) given by the flow map, the jump map, and the set of impulse times. These properties have been extensively studied and several sufficient conditions for asymptotic, input-to-state and integral input-to-state stability were obtained, even for systems with time-varying flow and jump maps and in the presence of time delays (see Hespanha et al., 2008; Chen & Zheng, 2009a, 2009b; Liu et al., 2011; Briat & Seuret, 2012; Wang et al., 2013; Dashkovskiy & Mironchenko, 2013; Liu et al., 2014; Gao & Wang, 2016; Barreira & Valls, 2016; Dashkovskiy & Feketa, 2017; Mancilla-Aguilar & Haimovich, 2020; Hong & Zhang, 2019; Feketa & Bajcinca, 2019, among others). Most of these references assume the existence of a Lyapunov-type function which may provide some degree of robustness with respect to the inclusion of disturbances or modeling errors. In addition, some of these also explicitly address robustness of stability.

This paper is concerned with a fundamental question: whether asymptotic stability of an impulsive system without disturbances or with zero input may guarantee some kind of robustness of the system with respect to the inclusion of inputs/disturbances.

For nonimpulsive (time-varying) systems under reasonable continuity assumptions on the function f defining the dynamics (local Lipschitz continuity, uniformly with respect to the time variable), the uniform asymptotic stability of the system when the input or the disturbance is identically zero (0-UAS) guarantees various kinds of robustness properties of the system with inputs/disturbances. For example, it is known that 0-UAS implies that the system with inputs/disturbances is totally stable (TS) (Hahn, 1967) which roughly speaking means that trajectories corresponding to small initial conditions and small inputs or disturbances remain near the equilibrium point. Another robustness property implied by the 0-UAS property is the so-called converging input converging state (CICS): every bounded trajectory which lies in the domain of attraction and corresponds to an input or disturbance that approaches zero must also converge to the equilibrium (Sontag, 2003; Ryan & Sontag, 2006; Mancilla-Aguilar & Haimovich, 2017).

☆ Partially supported by ANPCyT, Argentina, under grant PICT 2018-1385. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Björn S. Rüffer under the direction of Editor Daniel Liberzon.

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Yet another of these properties is captured by the fact that the combination of global uniform asymptotic stability under zero input/disturbance (0-GUAS) and a uniformly bounded-energy input/bounded state (UBEBS) property implies integral input-to-state stability (iISS, Sontag, 1998), as proved initially by Angeli et al. (2000) for time-invariant systems and extended to time-varying systems in Haimovich and Mancilla-Aguilar (2018b). A consequence of this fact can be loosely stated as follows: for some suitable way of measuring input energy, if the input energy is finite, then the state will converge to zero. This property can also be interpreted as providing some robustness of stability with respect to the inclusion of inputs, but provided that the state remains bounded under inputs of bounded energy.

In this paper, we address impulsive systems where both the flow and jump maps could be time-varying and depend on external inputs. We show that even under stronger uniform boundedness, and state and input Lipschitz continuity assumptions on the flow and jump maps, 0-GUAS implies neither CICS nor TS, and 0-GUAS and UBEBS do not imply iISS. We show that this is so even when 0-GUAS is uniform not only with respect to initial time but also over all possible impulse–time sequences and, moreover, also when the flow and jump maps are time-invariant and the former is input-independent. A very salient feature of our negative results is that they do not depend on how the input energy is measured; in other words, they are valid for any UBEBS and iISS gains, even of course when these could be different from each other. The results that we provide thus clearly illustrate that the stated nonrobustness of impulsive systems is of a very profound nature. This lack of robustness is directly related neither to how the input may enter into the system equations nor to the regularity of the flow and jump maps; it is indeed related to the fact that the definition of 0-GUAS usually considered in the literature of impulsive systems is too weak for guaranteeing any meaningful robustness property.

For (time-invariant) well-posed hybrid systems, it is known that 0-GUAS is robust with respect to the inclusion of small inputs (Cai & Teel, 2009, Prop. 2.4) or small model perturbations (Goebel et al., 2012, Thm. 7.21). For this to happen, however, stability must take hybrid time into account, thus causing decay towards the equilibrium set not only when continuous time elapses but also whenever jumps occur. In this regard, we have already shown that if, in addition to elapsed time, the number of jumps is taken into account in the definition of asymptotic stability, then 0-GUAS and UBEBS imply iISS (see Haimovich et al., 2019; Haimovich & Mancilla-Aguilar, 2020). The main contribution of this paper is thus to show that taking the number of jumps into account within the stability definition is unavoidable for the stated robustness to be possible.

Notation. $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, and $\mathbb{R}_{\geq 0}$ denote the natural numbers, the nonnegative integers, the reals, and the nonnegative reals, respectively. $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^p$ with $p \in \mathbb{N}$. We write $\alpha \in \mathcal{K}$ if $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous, strictly increasing and $\alpha(0) = 0$, and $\alpha \in \mathcal{K}_\infty$ if, in addition, α is unbounded. We write $\beta \in \mathcal{KL}$ if $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $\beta(\cdot, t) \in \mathcal{K}_\infty$ for any $t \geq 0$ and, for any fixed $r \geq 0$, $\beta(r, t)$ monotonically decreases to zero as $t \to \infty$. For $r \in \mathbb{R}$, $\lceil r \rceil$ and $\lfloor r \rfloor$ denote the least integer not less and the greatest integer not greater, respectively, than r.

2. Problem statement

Consider the time-varying impulsive system with inputs Σ defined by the equations

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad \text{for } t \notin \gamma, \tag{1a}$$

$$x(t) = h(t, x(t^-), u(t)), \qquad \text{for } t \in \gamma, \tag{1b}$$

where $t \geq 0$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, f and h are functions from $\mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$ such that $f(t, 0, 0) = 0$ and $h(t, 0, 0) = 0$ for all $t \geq 0$, and $\gamma = \{\tau_k\}_{k=1}^\infty$, with $0 < \tau_1 < \tau_2 < \cdots$ and $\lim_{k\to\infty} \tau_k = \infty$, is the impulse–time sequence. By "input", we mean a Lebesgue measurable and locally bounded function $u : [0, \infty) \to \mathbb{R}^m$; we denote by $\mathcal{U}$ the set of all the inputs. An input u could represent, e.g., a control input or a disturbance input. We define $\tau_0 := 0$.

A solution of Σ corresponding to an initial time $t_0 \geq 0$, an initial state $x_0 \in \mathbb{R}^n$ and an input $u \in \mathcal{U}$ is a right-continuous function $x : [t_0, T_x) \to \mathbb{R}^n$ such that $x(t_0) = x_0$ and:

(i) x is locally absolutely continuous on each nonempty interval J of the form $J = [\tau_k, \tau_{k+1}) \cap [t_0, T_x)$, with $k \geq 0$, and $\dot{x}(t) = f(t, x(t), u(t))$ for almost all $t \in J$; and
(ii) for all $\tau_k \in (t_0, T_x)$, the left limit $x(\tau_k^-)$ exists and is finite, and $x(\tau_k) = h(\tau_k, x(\tau_k^-), u(\tau_k))$.

The solution x is said to be maximally defined if no other solution $y : [t_0, T_y) \to \mathbb{R}^n$ satisfies $y(t) = x(t)$ for all $t \in [t_0, T_x)$ and has $T_y > T_x$. A solution x is forward complete if $T_x = \infty$, and Σ is forward complete if every maximal solution of Σ is forward complete.

We will use $\mathcal{T}(t_0, x_0, u)$ to denote the set of maximally defined solutions of Σ corresponding to initial time t_0, initial state x_0 and input u.

An important problem in control theory is understanding the dependence of state trajectories on the inputs, in particular when the inputs are bounded or when they converge to zero as $t \to \infty$. In order to make the latter precise, given an input $u \in \mathcal{U}$, an interval $I \subset \mathbb{R}_{\geq 0}$, and functions $\rho_1, \rho_2 \in \mathcal{K}_\infty$, we define

$$\|u_I\|_\infty := \max\left\{ \operatorname*{ess\,sup}_{t \in I} |u(t)|, \sup_{t \in \gamma \cap I} |u(t)| \right\}, \tag{2}$$

$$\|u_I\|_{\rho_1,\rho_2} := \int_I \rho_1(|u(s)|) ds + \sum_{s \in \gamma \cap I} \rho_2(|u(s)|). \tag{3}$$

When $I = (t_0, \infty)$ we simply write u_{t_0} instead of u_I. In both input bounds the values of u at the instants $t \in \gamma$ are explicitly taken into account, since these values may instantaneously affect the state trajectory.

The following stability properties give characterizations of the behavior of the state trajectories when the inputs are bounded, converge to zero, or are identically zero. In what follows, $\mathbf{0}$ denotes the identically zero input.

Definition 2.1. The impulsive system Σ is said to be:

(a) zero-input globally uniformly asymptotically stable (0-GUAS) if there exists $\beta \in \mathcal{KL}$ such that for all $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$, and $x \in \mathcal{T}(t_0, x_0, \mathbf{0})$, it happens that x is forward complete and for all $t \geq t_0$

$$|x(t)| \leq \beta(|x_0|, t - t_0); \tag{4}$$

(b) uniformly bounded-energy input/bounded state (UBEBS) if Σ is forward complete and there exist $\alpha, \rho_1, \rho_2 \in \mathcal{K}_\infty$ and $c \geq 0$ such that

$$\alpha(|x(t)|) \leq |x_0| + \|u_{(t_0,t]}\|_{\rho_1,\rho_2} + c \tag{5}$$

for all $t \geq t_0 \geq 0$, $x_0 \in \mathbb{R}^n$, $u \in \mathcal{U}$, and $x \in \mathcal{T}(t_0, x_0, u)$;

(c) integral input-to-state stable (iISS) if Σ is forward complete and there exist $\beta \in \mathcal{KL}$ and $\alpha, \rho_1, \rho_2 \in \mathcal{K}_\infty$ such that for all $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$, $u \in \mathcal{U}$ and $x \in \mathcal{T}(t_0, x_0, u)$, it happens that for all $t \geq t_0$,

$$\alpha(|x(t)|) \leq \beta(|x_0|, t - t_0) + \|u_{(t_0,t]}\|_{\rho_1,\rho_2}; \tag{6}$$

(d) converging-input converging-state (CICS) if every forward complete and bounded solution $x \in \mathcal{T}(t_0, x_0, u)$, with $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$, and $u \in \mathcal{U}$ such that $\|u_t\|_\infty \to 0$ as $t \to \infty$, satisfies $x(t) \to 0$ as $t \to \infty$;

(e) totally stable (TS) if $f(t, \xi, \mu) \equiv f_0(t, \xi) + \mu$, $h(t, \xi, \mu) \equiv h_0(t, \xi) + \mu$ and, for every $\varepsilon > 0$ there exists $\delta > 0$ such that every solution $x \in \mathcal{T}(t_0, x_0, u)$, with $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ with $|x_0| < \delta$, and $u \in \mathcal{U}$ such that $\|u_{t_0}\|_\infty < \delta$, satisfies $|x(t)| < \varepsilon$ for all $t \in [t_0, T_x)$.

Remark 2.2. The definition of total stability given above is a natural generalization, to impulsive systems, of the one usually considered in the literature of ordinary differential equations (see Hahn, 1967, Chapter VII). ∘

Definition 2.3. We say that (1) is 0-GUAS (or UBEBS) uniformly with respect to a set $\mathcal{S}$ of impulse–time sequences if every system Σ defined by (1) with $\gamma \in \mathcal{S}$ is 0-GUAS (or UBEBS) and, moreover, the bound (4) [or (5)] holds with the same β (or ρ_1, ρ_2, α and c) for every such system.

From the definition of these stability properties, it easily follows that iISS implies 0-GUAS and UBEBS. For nonimpulsive time-varying systems and under appropriate assumptions on the flow map f, it was proved that 0-GUAS and UBEBS imply iISS (Haimovich & Mancilla-Aguilar, 2018b, Theorem 1), and that 0-GUAS implies TS (Hahn, 1967, Theorem 56.4) and CICS (Mancilla-Aguilar & Haimovich, 2017, Section 3.2). The question that naturally arises is thus whether the same implications remain true for impulsive systems.

In Haimovich and Mancilla-Aguilar (2018a, Theorem 3.2) it was shown that 0-GUAS and UBEBS imply iISS for time-varying impulsive systems, assuming that the impulse–time sequence satisfies the so-called uniform incremental boundedness (UIB) condition (Haimovich & Mancilla-Aguilar, 2018a, Definition 3.2), and in Haimovich et al. (2019) that the same implication holds without the UIB condition if the 0-GUAS property is strengthened by ensuring that decay towards the equilibrium occurs not only as time advances but also as jumps occur, i.e. by replacing $\beta(|x_0|, t - t_0)$ by $\beta(|x_0|, t - t_0 + n_{(t_0,t]})$ in (4), where $n_{(t_0,t]}$ is the number of impulse times lying in $(t_0, t]$.

The main result of the current paper is to show that the mentioned implications do not remain valid if 0-GUAS is understood in the usual sense and the UIB condition is not assumed. We will do so through counterexamples in the next section.

3. Main results

In this section, we show that even if the flow and jump maps are time-invariant, 0-GUAS implies neither CICS nor TS, and 0-GUAS and UBEBS do not imply iISS. In addition, we will show that these negative results remain true when 0-GUAS and UBEBS are uniform over all impulse–time sequences (as per Definition 2.3) but the jump map is allowed to be time-varying.

In Section 3.1, we give the base equations for our counterexamples, jointly with some illustration of the functions involved. In Section 3.2, we provide a conceptual explanation for the rationale of our main results. The technical proof for the 0-GUAS and UBEBS properties is given in Section 3.3. In Section 3.4, we show that 0-GUAS implies neither CICS nor TS and in Section 3.5, that 0-GUAS and UBEBS do not imply iISS.

3.1. Impulsive system equations

Consider the scalar impulsive system Σ, with a single input, of the form (1) with

$$f(t, \xi, \mu) = -\xi, \tag{7}$$

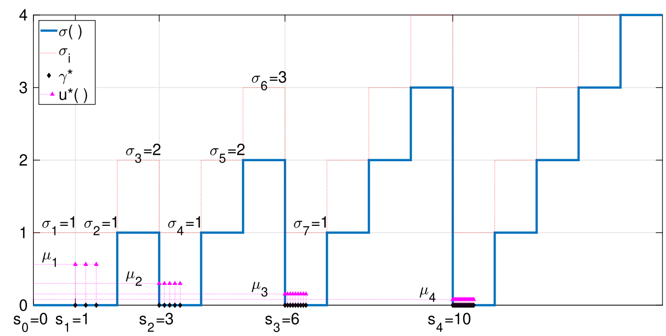

Fig. 1. Solid blue: the function σ; dashed red: first values of $\{\sigma_k\}$; black diamonds: first values of the sequence γ^* defined in Section 3.4; magenta triangles: the input $u^*(\cdot)$ defined in (14)–(15).

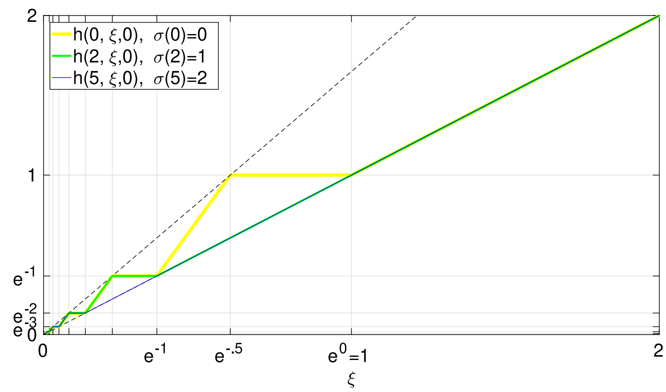

Fig. 2. The function $\xi \mapsto h(t, \xi, 0)$ for $\xi \geq 0$. Yellow: $t = 0$, $\sigma(t) = 0$; green: $t = 2$, $\sigma(t) = 1$; blue: $t = 5$, $\sigma(t) = 2$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$h(t, \xi, \mu) = \hat{h}(t, \xi) + \mu, \tag{8}$$

$$\hat{h}(t, \xi) = \begin{cases} \bar{h}(|\xi|) & \text{if } |\xi| \leq e^{-\sigma(t)}, \\ |\xi| & \text{otherwise}, \end{cases} \tag{9}$$

where $\bar{h} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is defined by

$$\bar{h}(r) = \begin{cases} 0 & \text{if } r = 0, \\ e^{\lceil \ln r \rceil} & \text{if } \lceil \ln r \rceil - 0.5 < \ln r \leq 0, \\ (1 + e^{0.5})r - e^{\lceil \ln r \rceil - 0.5} & \text{if } \ln r \leq \lceil \ln r \rceil - 0.5 \leq 0, \\ r & \text{otherwise}, \end{cases} \tag{10}$$

and σ is defined as follows. For $i \in \mathbb{N}$, let S_i be the increasing and finite sequence containing the first i natural numbers, i.e. $S_1 = \{1\}$, $S_2 = \{1, 2\}$, etc., and construct the infinite sequence $\{\sigma_k\}_{k=1}^\infty$ by concatenating $S_1, S_2, \ldots$ Then, define

$$\sigma(t) := \sigma_i - 1 \quad \text{when } t \in [i - 1, i), \ i \in \mathbb{N}.$$

The function σ can be equivalently defined as follows:

$$\sigma(t) = i, \quad \text{for } t \in [s_j + i, s_j + i + 1), \qquad i = 0, \ldots, j, \tag{11}$$

$$s_j := \sum_{i=0}^{j} i, \qquad j \in \mathbb{N}_0. \tag{12}$$

The function $\sigma(\cdot)$ is illustrated in Fig. 1. The map h depends on t only through the function σ. Fig. 2 illustrates the function $\xi \mapsto h(t, \xi, 0)$ for $\xi \geq 0$ and for $t = 0$, 2 and 5, corresponding to $\sigma(t)$ assuming the values 0, 1 and 2, respectively. The function h in (8) is nondecreasing in the absolute value of the state variable when the other variables are fixed, i.e. h satisfies $h(t, |\xi_1|, \mu) \geq$

$h(t, |\xi_2|, \mu)$ for all $t \geq 0$, $\mu \in \mathbb{R}$ and $|\xi_1| \geq |\xi_2| \geq 0$. Note that system Σ with f and h defined, respectively, by (7) and (8), and γ any impulse–time sequence, is forward complete, since the differential equation (1a) has no finite escape time.

Remark 3.1. If the impulse–time sequence γ is such that $\sigma(t) = 0$ for every $t \in \gamma$, then the evolution of (1) with (7)–(8) becomes equivalent to that arising when h in (8) is replaced by the time-invariant jump map $h_{\mathrm{ti}}(\xi, \mu) = h(0, \xi, \mu)$. Therefore, in such a case the system (1) with (7)–(8) is equivalent to an impulsive system having time-invariant flow and jump maps. ∘

3.2. Possible system behaviors

We next provide some insight into the features that an impulsive system of the form (1) with f and h as given in Section 3.1 may have. Note that the impulse–time sequence γ must be specified for the system to be fully defined. Nonetheless, we will explain some general features that may be present for suitable impulse–time sequences.

First, note that the flow map f in (7) is linear, time-invariant and input-independent, and hence the continuous part of the solution, i.e. between jumps, is a decaying exponential.

3.2.1. The zero-input system

Next, consider the map h in (8)–(10) under zero input. Note that after the first jump, the state will become nonnegative and, taking the flow equation (7) also into account, remain nonnegative thereafter. We may thus focus only on nonnegative values of the state. Since the map h depends on t only through $\sigma(t)$, then the yellow graph in Fig. 2 illustrates $h(t, \cdot, 0)$ not only at $t = 0$ but also at every value of t for which $\sigma(t) = \sigma(0) = 0$. As follows from (11) and shown in Fig. 1, $\sigma(t) = 0$ for all $t \in [s_j, s_j{+}1)$, for all $j \in \mathbb{N}_0$. Fig. 2 shows that the after-jump value of the state at such values of t may be equal to or greater than the value immediately before the jump. Therefore, the occurrence of a jump may have a destabilizing effect, depending on the state value. For example, for $\xi = e^{-.5} \approx 0.6$, we have $h(t, \xi, 0) = 1 > \xi$ if t is such that $\sigma(t) = 0$.

The map $h(0, \cdot, 0) = \bar{h}(\cdot)$, which equals $h(t, \cdot, 0)$ for t such that $\sigma(t) = 0$, also has the following properties: for all $k \in \mathbb{N}$,

$$h^k(0, \xi, 0) = \bar{h}^k(\xi) = \underbrace{\bar{h} \circ \cdots \circ \bar{h}}_{k \text{ times}}(\xi) \leq e^{\lceil \ln \xi \rceil},$$

and for some $k_0 = k_0(\xi)$, $h^{k_0}(0, \xi, 0) = e^{\lceil \ln \xi \rceil}$.

This means that no matter how many times we may iterate $h(0, \cdot, 0)$, the resulting value will be bounded and hence the mere occurrence of an infinite number of successive impulses (under zero input) cannot make the state diverge.

Irrespective of the initial time t_0, the function σ is such that if we wait enough time, some time period will arrive during which $\sigma(t)$ is as large as desired and hence $e^{-\sigma(t)}$ is as small as desired. During this time period, jumps will leave the state unaltered and hence the flow equation will make the state decay exponentially towards the origin. This will happen until the value $\sigma(t)$ becomes small again, and hence $e^{-\sigma(t)}$ becomes larger, allowing jumps to shift the state away from the origin but only in a bounded manner, as previously explained. Then again, after waiting enough time jumps will eventually become 'harmless' and the flow equation will drive the state even closer to the origin, and so on. This is the rationale for the 0-GUAS property (see also Example 1 in Section 3.5).

3.2.2. The system with nonzero input

We already know that during flows the state decays exponentially towards the origin and that, under zero input, not even an infinite succession of jumps can make the state diverge. However, the occurrence of an arbitrarily small input can severely modify this stable behavior. Take for example $\xi = e^{-1}$ for which $\xi = h(t, \xi, 0)$ for all t and hence the successive application of $h(t, \cdot, 0)$ yields $h^k(t, \xi, 0) = \xi$ for all $k \in \mathbb{N}$. For an input value $\mu = \delta > 0$, it follows that $\zeta := h(t, \xi, \mu) > \xi$ for all t such that $\sigma(t) = 0$. After this, the occurrence, on an interval in which $\sigma(t) = 0$, of many jumps in an appropriate rapid succession under zero input will cause the state after the kth jump, with k large enough, to be approximately $h^k(t, \zeta, 0) \simeq e^{\lceil \ln \zeta \rceil} = e^0 = 1$. Therefore, the behavior arising from the initial state $\xi = e^{-1}$, which under zero input just remains bounded by e^{-1}, can be drastically modified to almost reach $e^0 = 1$ by an arbitrarily small input. This is the rationale for the lack of stability with respect to the inclusion of inputs, a feature that prevents the CICS and TS properties. However, since jumps are harmless for state values greater than 1, and inputs enter additively into (8), the state cannot diverge under inputs of bounded energy and hence the UBEBS property holds.

3.3. 0-GUAS and UBEBS

The following property of σ is the precise formulation of the fact that if we wait enough time, some time period will arrive during which $\sigma(t)$ is as large as desired. This will be instrumental in establishing 0-GUAS.

Lemma 3.2. *For every $k \in \mathbb{N}_0$ there exists $\bar{T}_k > 0$ such that for every $t_0 \geq 0$ there exists $t^* \in [t_0, t_0 + \bar{T}_k]$ such that $\sigma(s) \geq k + 1$ for all $s \in [t^*, t^* + 1]$.*

Proof. Let $k \in \mathbb{N}_0$. Set $\bar{T}_k = k + 1 + s_{k+2}$, with $s_j = \sum_{i=0}^{j} i$. According to (11), we have $\sigma(s) \geq k + 1$ for $s \in [s_j + k + 1, s_{j+1})$ for all $j \geq k + 1$.

If $t_0 \leq \bar{T}_k$, take $t^* = \bar{T}_k$.

If $t_0 > \bar{T}_k$, let $\kappa := \max\{j \in \mathbb{N}_0 : s_j \leq t_0\}$. Clearly, $\kappa \geq k + 2$ and $s_\kappa \leq t_0 < s_{\kappa+1}$. If $s_\kappa \leq t_0 \leq s_\kappa + k + 1$, take $t^* = s_\kappa + k + 1$.

If $s_\kappa + k + 1 < t_0 \leq s_{\kappa+1} - 1$, take $t^* = t_0$.

Otherwise, take $t^* = s_{\kappa+1} + k + 1$. ■

In words, Lemma 3.2 establishes that for every value $k \in \mathbb{N}_0$, no matter how large, there exists a maximum time period $\bar{T}_k$ with the following property: given any possible initial time t_0, the function σ will be not less than $k + 1$ for a whole time unit in a time period contained within the interval $[t_0, t_0 + \bar{T}_k]$. One feature that will allow to establish 0-GUAS is the fact that $\bar{T}_k$ does not depend on the initial time t_0.

Lemma 3.3. *The impulsive system Σ in (1), with f and h defined by (7) and (8), respectively, and γ any impulse–time sequence, is 0-GUAS and UBEBS. Moreover, (1) with (7)–(8) is 0-GUAS and UBEBS both uniformly with respect to the set of all impulse–time sequences.*

Proof. We will prove that Σ is 0-GUAS by establishing first that: (i) Σ is 0-input globally uniformly stable (0-GUS), i.e. there exists $\nu \in \mathcal{K}_\infty$ such that for all $x \in \mathcal{T}(t_0, x_0, \mathbf{0})$ with $t_0 \geq 0$ and $x_0 \in \mathbb{R}$ it happens that $|x(t)| \leq \nu(|x_0|)$ for all $t \geq t_0$; and (ii) for all $0 < \varepsilon < r$ there exists $T = T(r, \varepsilon) \geq 0$ such that for all $x \in \mathcal{T}(t_0, x_0, \mathbf{0})$ with $t_0 \geq 0$ and $|x_0| \leq r$ we have that $|x(t)| \leq \varepsilon$ for some $t \in [t_0, t_0 + T]$.

Since for all $x \in \mathcal{T}(t_0, x_0, \mathbf{0})$, $|x| \in \mathcal{T}(t_0, |x_0|, \mathbf{0})$, and $x(t) \equiv 0$ when $x_0 = 0$, we only have to establish (i) and (ii) for positive initial conditions x_0. Let $x \in \mathcal{T}(t_0, x_0, \mathbf{0})$ with $t_0 \geq 0$ and $x_0 > 0$.

Then $x(t) > 0$ for all $t \geq t_0$. If $x_0 > 1$ then $x(t) \leq x_0$ for all $t \geq t_0$. Suppose that the latter is not true. Since $x(t)$ is nonincreasing between consecutive impulse times, the first time t for which $x(t) > x_0$ must be an impulse time $t \in \gamma \cap (t_0, \infty)$. For such a time t we have that $x(t^-) \leq x_0$ and $x(t) = h(t, x(t^-), 0)$. Since h is nondecreasing in the absolute value of its second argument, we have that $x(t) = h(t, x(t^-), 0) \leq h(t, x_0, 0) = x_0$, which is absurd. Suppose now that $0 < x_0 \leq 1$. Let $k(x_0) = \lceil \ln x_0 \rceil \leq 0$. Then $x(t) \leq e^{k(x_0)}$ for all $t \geq t_0$. Suppose on the contrary that $x(t) > e^{k(x_0)}$ for some $t \geq t_0$. Since x is nonincreasing between consecutive impulse–times and $x(t_0) \leq e^{k(x_0)}$, the first time $t \geq t_0$ for which $x(t) > e^{k(x_0)}$ has to be an impulse–time $t \in \gamma \cap (t_0, \infty)$. Then we have that $x(s) \leq e^{k(x_0)}$ for all $s \in [t_0, t)$ and $x(t) = h(t, x(t^-), 0) > e^{k(x_0)}$. Since $x(t^-) \leq e^{k(x_0)}$ and h is nondecreasing in the absolute value of its second argument, we have that $h(t, x(t^-), 0) \leq h(t, e^{k(x_0)}, 0) = e^{k(x_0)}$, arriving to a contradiction.

Since the function $\bar{\nu}(r) = r$ for $r > 1$ and $\bar{\nu}(r) = e^{\lceil \ln r \rceil}$ for $0 < r \leq 1$ is non decreasing and $\lim_{r \to 0^+} \bar{\nu}(r) = 0$, there exists $\nu \in \mathcal{K}_\infty$ such that $\bar{\nu}(r) \leq \nu(r)$ for all $r \geq 0$ (see, e.g. Clarke et al., 1998, Lemma 2.5). In consequence, Σ satisfies item (i) with such a function ν.

For establishing (ii) we first prove the following.

Claim 1. *Let $k \in \mathbb{N}_0$. Then there exists $T_k > 0$ such that for all $x \in \mathcal{T}(t_0, x_0, \mathbf{0})$ with $t_0 \geq 0$ and $0 < x_0 \leq e^{-k}$ there is a $t \in [t_0, t_0 + T_k]$ such that $x(t) \leq e^{-(k+1)}$.*

Proof of Claim 1. Suppose that $x \in \mathcal{T}(t_0, x_0, \mathbf{0})$ with $t_0 \geq 0$ and $0 < x_0 \leq e^{-k}$. Note that $0 < x(t) \leq e^{k(x_0)} \leq e^{-k}$ for all $t \geq t_0$, with $k(x_0)$ as defined above. Let $\bar{T}_k > 0$ and $t^* \in [t_0, t_0 + \bar{T}_k]$ be the quantities coming from Lemma 3.2. Suppose that $x(t) > e^{-(k+1)}$ for all $t \in [t^*, t^* + 1]$. Since $\sigma(t) \geq k + 1$ on $[t^*, t^* + 1]$, from the definitions of f and h we have that $x(t) = x(t^*)e^{-(t-t^*)}$ for all $t \in [t^*, t^* + 1]$. Therefore $x(t^* + 1) = x(t^*)e^{-1} \leq e^{-(k+1)}$, which is absurd. In consequence there exists $t \in [t^*, t^* + 1]$ so that $x(t) \leq e^{-(k+1)}$, and the claim follows by taking $T_k = \bar{T}_k + 1$. ◦

We proceed to prove (ii). Let $0 < \varepsilon < r$. Suppose that $r \leq 1$. Let $t_0 \geq 0$, $0 < x_0 \leq r$ and $x \in \mathcal{T}(t_0, x_0, \mathbf{0})$. Pick $k_1, k_2 \in \mathbb{N}_0$ such that $e^{-k_2} < \varepsilon < r \leq e^{-k_1}$. Let $T_{k_1}, T_{k_1+1}, \ldots, T_{k_2-1}$ be the quantities coming from Claim 1 corresponding to $k = k_1, \ldots, k_2 - 1$. Applying Claim 1 in a recursive way, it follows that there exists $t_0 \leq t_1 \leq \cdots \leq t_{k_2-k_1}$, with $t_1 - t_0 \leq T_{k_1}, \ldots, t_{k_2-k_1} - t_{k_2-k_1-1} \leq T_{k_2-1}$ such that $x(t_j) \leq e^{-(k_1+j)}$ for all $j = 1, \ldots, k_2 - k_1$. In consequence $x(t_{k_2-k_1}) \leq e^{-k_2} < \varepsilon$ and $t_{k_2-k_1} - t_0 \leq \sum_{i=0}^{k_2-k_1-1} T_{k_1+j}$. So item (ii) holds in this case with $T(r, \varepsilon) = \sum_{i=0}^{k_2-k_1-1} T_{k_1+j}$.

Suppose now that $r > 1$. If $x_0 \leq 1$, then $x(t) < \varepsilon$ for some $t \in [t_0, t_0 + T(1, \varepsilon)]$. If $x_0 > 1$, then by solving Eqs. (1) on the interval $I = [t_0, t_0 + \ln x_0]$ it follows that $x(t) = x_0 e^{-(t-t_0)}$ for all $t \in I$. So $x(t_0 + \ln x_0) = 1$. In consequence, there exists $t \in [t_0 + \ln x_0, t_0 + \ln x_0 + T(1, \varepsilon)]$ such that $x(t) < \varepsilon$, and (ii) follows with $T(r, \varepsilon) = \ln r + T(1, \varepsilon)$ in this case.

We next prove that: (iii) for all $0 < \varepsilon < r$ there exists $T^* = T^*(r, \varepsilon) \geq 0$ such that for all $x \in \mathcal{T}(t_0, x_0, \mathbf{0})$ with $t_0 \geq 0$ and $|x_0| \leq r$ we have that $|x(t)| \leq \varepsilon$ for all $t \geq t_0 + T^*$. Let $\hat{\varepsilon} = \nu^{-1}(\varepsilon)$ and $T^* = T(r, \hat{\varepsilon})$, with ν and $T(\cdot, \cdot)$ as in (i) and (ii) respectively. Then, there exists $t^* \in [t_0, t_0 + T^*]$ such that $|x(t^*)| \leq \hat{\varepsilon}$. Due to (i), $|x(t)| \leq \nu(\hat{\varepsilon}) = \varepsilon$ for all $t \geq t^*$ and therefore for all $t \geq t_0 + T^*$.

The existence of a function $\beta \in \mathcal{KL}$ as in the definition of 0-GUAS follows from (i) and (iii) and the steps used in the proof of Lin et al. (1996, Proposition 2.5). The fact that the same β can be used for every impulse–time sequence γ follows from the fact that neither the function ν in (i) nor the time $T(r, \varepsilon)$ in (ii) depends on the specific γ.

The system Σ is UBEBS because for all $x \in \mathcal{T}(t_0, x_0, u)$ with $t_0 \geq 0$, $x_0 \in \mathbb{R}$ and $u \in \mathcal{U}$,

$$|x(t)| \leq |x_0| + \|u_{(t_0,t]}\| + 1 \tag{13}$$

holds for $\|u_I\|$ defined as

$$\|u_I\| = \int_I |u(t)| dt + \sum_{t \in \gamma \cap I} |u(t)|.$$

For a contradiction, suppose there is $t \geq 0$ such that (13) does not hold. Since $|x|$ is nonincreasing between consecutive impulse times, the first time t^* for which (13) is not true must satisfy $t^* \in \gamma \cap (t_0, \infty)$. Then $|x(t)| \leq |x_0| + \|u_{(t_0,t]}\| + 1$ for all $t \in [t_0, t^*)$ and $|x(t^*)| > |x_0| + \|u_{(t_0,t^*]}\| + 1$. Since $|x(t^{*-})| \leq |x_0| + \|u_{(t_0,t^*)}\| + 1$, it follows that

$$\begin{aligned} |x(t^*)| &\leq h(t^*, |x(t^{*-})|, u(t^*)) \\ &\leq h(t^*, |x_0| + \|u_{(t_0,t^*)}\| + 1, u(t^*)) \\ &\leq |x_0| + \|u_{(t_0,t^*)}\| + 1 + |u(t^*)| = |x_0| + \|u_{(t_0,t^*]}\| + 1, \end{aligned}$$

which is absurd. Here we have used the facts that h is nondecreasing in its second argument and that $h(t, \xi, \mu) = |\xi| + \mu$ if $|\xi| \geq 1$. Since (13) holds for every impulse–time sequence γ, then we have also established UBEBS uniformly with respect to the set of all impulse–time sequences. ■

Remark 3.4. The functions f and $g(t, \xi, \mu) := h(t, \xi, \mu) - \xi$ belong to $\mathcal{AL}$ with the class $\mathcal{AL}$ as defined in Haimovich and Mancilla-Aguilar (2018a, Definition 3.1). So, Σ is iISS for every impulse–time sequence γ which is UIB according to Haimovich and Mancilla-Aguilar (2018a, Theorem 3.2). See also Haimovich et al. (2019). ◦

3.4. 0-GUAS implies neither CICS nor TS

We next show that for the impulse–time sequence γ^* defined below, the system Σ with f and h given by (7)–(8) is not CICS and that if we replace (7) by $f(t, \xi, \mu) = -\xi + \mu$, then the resulting system Σ is not TS.

For $N \in \mathbb{N}$, consider the finite sequence $\mathcal{S}_N = \{\tau_{N,k}\}_{k=0}^{2^N}$ with $\tau_{N,k} = s_N + \frac{k}{2^{N+1}}$, where s_N is defined by (12). Define γ^* as the sequence obtained by concatenating $\mathcal{S}_1, \mathcal{S}_2, \ldots$, so that the first elements of γ^* are $s_1, s_1 + \frac{1}{4}, s_1 + \frac{2}{4}, s_2, s_2 + \frac{1}{8}, s_2 + \frac{2}{8}, s_2 + \frac{3}{8}, s_2 + \frac{4}{8}, \ldots$ The first values of the sequence γ^* are plotted as black diamonds in Fig. 1. This sequence contains $2^N + 1$ elements in every interval $[s_N, s_N + 1/2]$ of length $1/2$. Therefore, although γ^* has no finite limit points, it contains an increasing number of elements which become closer and closer together within intervals of fixed length $1/2$ as time progresses. The elements of the sequence γ^* are placed so that $\sigma(t) = 0$ for every $t \in \gamma^*$ [recall (11)–(12) and see Fig. 1]. According to Remark 3.1, the system Σ in (1) with (7)–(8) and $\gamma = \gamma^*$ becomes equivalent to a system with time-invariant flow and jump maps.

Theorem 3.5. *The system Σ with f and h given by (7)–(8) and $\gamma = \gamma^*$ is not CICS.*

Proof. Consider the input u^* defined as follows:

$$u^*(t) = \begin{cases} \mu_N & \text{if } t = \tau_{N,k} : N \in \mathbb{N}, k = 0, \ldots, 2^N, \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

$$\mu_N = \frac{1 - e^{-\Delta_N}}{1 - e^{-1/2}}, \qquad \Delta_N = 2^{-(N+1)}. \tag{15}$$

The input u^* is illustrated in magenta in Fig. 1. Note that $\|u^*_{s_N}\|_\infty = \|u^*_{(t_0,\infty)}\|_\infty = \mu_N$ for all N [recall (2)], since $|u^*(t)| \leq \mu_N$ for all $t > s_N$ and $u^*(\tau_{N,k}) = \mu_N$ for all $k = 0, \ldots, 2^N$. Therefore

$\|u_t^*\|_\infty \to 0$ as $t \to \infty$ because $\|u_t^*\|$ is nonincreasing in t and $\mu_N \to 0$.

Let x be the unique solution of Σ corresponding to initial time 0, initial state 0 and input u^*. We claim that $x(s_N + 1/2) \geq 1$ for all $N \in \mathbb{N}$.

By solving the equations of Σ on $[0, s_1]$ it is clear that $x(s_1) = \mu_1 > 0$. So, $x(t) > 0$ for all $t \geq s_1 = 1$. Let $N \in \mathbb{N}$, then, if $I_N = [s_N, s_N + 1/2]$, we have that $\gamma^* \cap I_N = \mathcal{S}_N$ and that $\sigma(t) = 0$ for all $t \in I_N$. Let $\xi_{N,k} = x(\tau_{N,k})$ for $k = 0, \dots, 2^N$. Then, for all $0 \leq k \leq 2^N - 1$, $\xi_{N,k+1} = \bar{h}(\xi_{N,k} e^{-\Delta_N}) + \mu_N$. Since $\bar{h}(r) \geq r$ for all $r \geq 0$, we have that for all $1 \leq k \leq 2^N - 1$, then $\xi_{N,k+1} \geq \xi_{N,k} e^{-\Delta_N} + \mu_N$. By induction on k it can be proved that $\xi_{N,k} \geq \mu_N \sum_{j=0}^{k-1} e^{-j\Delta_N}$. Therefore,

$$x(s_N + 1/2) = \xi_{N,2^N} \geq \mu_N \sum_{j=0}^{2^N-1} e^{-j\Delta_N} = 1.$$

Since $s_N \to \infty$, it follows that $x(t)$ does not converge to 0 as $t \to \infty$, and thus Σ is not CICS. ■

Theorem 3.6. *The system Σ with $f(t, \xi, \mu) = -\xi + \mu$, h given by (8) and $\gamma = \gamma^*$, is 0-GUAS but not TS.*

Proof. It is clear that Σ is 0-GUAS, since its zero-input system is the same as that of the system considered in Lemma 3.3. Consider the input u^* defined in (14)–(15) in the proof of Theorem 3.5. Since u^* is zero during flows, then u^* does not affect the exponential decay towards the origin that occurs during flows. Given $\delta > 0$, let N be so that $\mu_N < \delta$. Then $\|u^*_{s_N}\|_\infty < \delta$. Let $x \in \mathcal{T}(s_N, 0, u^*)$. By proceeding as in the proof of Theorem 3.5 it follows that $|x(s_N + 1/2)| \geq 1$, showing that Σ is not TS. ■

3.5. 0-GUAS and UBEBS do not imply iISS

Next, we prove that the same system considered in Theorem 3.5 is not iISS.

Theorem 3.7. *The system Σ with f and h given by (7)–(8) and $\gamma = \gamma^*$ given in Section 3.4 is not iISS.*

Theorem 3.7 is a straightforward consequence of the following result, which shows that irrespective of the initial time and of how small the input energy and initial state may be, we can always find an input that causes the state to become larger than a fixed value at some future time.

Lemma 3.8. *Consider the system Σ with f and h given by (7)–(8) and $\gamma = \gamma^*$ given in Section 3.4. Let $\rho_1, \rho_2 \in \mathcal{K}_\infty$ and define $\|u\| := \|u\|_{\rho_1,\rho_2}$ [recall (3)]. Let $\delta_1, \delta_2 > 0$. Then, there exist an initial time t_0, a time instant t, an initial state x_0 and an input u such that $0 \leq t_0 \leq t$, $|x_0| \leq \delta_1$, $\|u\| \leq \delta_2$, and the system solution x corresponding to t_0, x_0 and u satisfies $|x(t)| \geq e^{-1}$.*

Proof. If $\delta_1 \geq e^{-1}$, then the result follows trivially with arbitrary $t = t_0 \geq 0$ and $x_0 = \delta_1$. So, consider $\delta_1 < e^{-1}$. Define

$$n_0 = -\lfloor \ln \delta_1 \rfloor, \quad \bar{\mu} = \min\left\{\rho_2^{-1}(\delta_2/n_0), e^{-n_0+1} - e^{-n_0}\right\},$$
$$\Delta = \min\left\{\ln \frac{1}{1-\bar{\mu}}, \ln \frac{1+\sqrt{e}\bar{\mu}}{1+\bar{\mu}}\right\}.$$

Note that $0 < \Delta < 1/2$. Consider the input construction algorithm given as Algorithm 1. The rationale for this algorithm is as follows. It will later be shown that this algorithm generates a sequence of state values (namely $\{\xi_k\}$) that will constitute a lower bound for the state evolution at the impulse times. This input construction algorithm generates a zero input (namely $\mu_k = 0$) whenever the unforced system dynamics pushes the state farther from the origin. A small input ($\mu_k = \bar{\mu}$) is generated only when necessary to make the subsequent unforced dynamics keep steering the state farther, as explained conceptually in Section 3.2.2.

The algorithm begins by setting an initial condition for the state lower bound sequence $\xi_0 = e^{\lfloor \ln \delta_1 \rfloor} \leq \delta_1$, at the initialization step (I). Then, in the repeat block at (R1), it happens that $\ell_0 = -\lceil \ln \xi_0 - \Delta \rceil = -\lceil \lfloor \ln \delta_1 \rfloor - \Delta \rceil = -\lfloor \ln \delta_1 \rfloor = n_0$. At (R2), k is set to $k = 1$. Then, the if condition initially holds, because $-\ell_0 = \lceil \ln \xi_0 - \Delta \rceil \geq \ln \xi_0 - \Delta = \lfloor \ln \delta_1 \rfloor - \Delta = -\ell_0 - \Delta > -\ell_0 - 0.5$. Consequently, at the first iteration, corresponding to $k = 1$, (Ri1) to (Ri3) will be executed so that $\mu_1 = \bar{\mu}$, i is set to $i = 1$ and $k_1 = k = 1$. At (R3), we have $\xi_1 = \bar{h}(\xi_0 e^{-\Delta}) + \mu_1 = \bar{h}(e^{\lfloor \ln \delta_1 \rfloor - \Delta}) + \bar{\mu} = e^{\lfloor \ln \delta_1 \rfloor} + \bar{\mu}$, where we have used (10). Recalling the definition of $\bar{\mu}$, it follows that $\xi_0 < \xi_1 \leq e^{-n_0+1}$.

Algorithm 1: Input sequence construction

Data: δ_1, Δ, $\bar{\mu}$
Output: F, $\{\xi_k\}_{k=1}^F$, $\{\mu_k\}_{k=1}^F$, $\{k_i\}_{i=1}^{n_1}$

```
begin Initialization
    ξ_0 = e^{⌊ln δ_1⌋}, k ← 0, i ← 0;                        (I)
repeat
    ℓ_k = −⌈ln ξ_k − Δ⌉;                                       (R1)
    k ← k + 1;                                                  (R2)
    if −ℓ_{k−1} − 0.5 ≤ ln ξ_{k−1} − Δ ≤ −ℓ_{k−1} then
        μ_k = μ̄;                                               (Ri1)
        i ← i + 1;                                              (Ri2)
        k_i = k;                                                (Ri3)
    else
        μ_k = 0;                                                (Re)
    ξ_k = h̄(ξ_{k−1} e^{−Δ}) + μ_k;                              (R3)
until ξ_k ≥ e^{−1};
```

We claim that this algorithm finishes in a finite number of steps F that depends on δ_1 and δ_2, and that n_1, the number of iterations at which $\mu_k \neq 0$, satisfies $n_1 \leq n_0$ so that $\sum_{k=1}^F \rho_2(\mu_k) \leq n_1 \rho_2(\bar{\mu}) \leq \delta_2$. Whenever $-\ell_{k-1} - 0.5 \leq \ln \xi_{k-1} - \Delta \leq -\ell_{k-1}$ (this holds for $k = 1$), then according to (Ri1) and (R3) in Algorithm 1, and (10), then

$$\xi_k = e^{-\ell_{k-1}} + \bar{\mu} \geq \xi_{k-1} e^{-\Delta} + \bar{\mu} \geq \xi_{k-1}(1 - \bar{\mu}) + \bar{\mu}$$
$$= \xi_{k-1} + (1 - \xi_{k-1})\bar{\mu} \geq \xi_{k-1} + (1 - e^{-1})\bar{\mu} > \xi_{k-1} \tag{16}$$

provided $\xi_{k-1} \leq e^{-1}$ (otherwise, the algorithm would have stopped). Hence, $\ell_k \leq \ell_{k-1}$. Also in this case, we have

$$\xi_k e^{-\Delta} = (e^{-\ell_{k-1}} + \bar{\mu}) e^{-\Delta} \geq (e^{-\ell_{k-1}} + \bar{\mu}) \frac{1 + \bar{\mu}}{1 + \sqrt{e}\bar{\mu}}. \tag{17}$$

The function $\phi(r) = \frac{1+r}{1+\sqrt{e}r}$ is strictly decreasing in $\mathbb{R}_{\geq 0}$ and therefore $\phi(\bar{\mu}) > \phi(a\bar{\mu})$ for every $a > 1$. Take $a = e^{\ell_{k-1} - 0.5}$, which satisfies $a > 1$ because $\ell_{k-1} \geq 1$ whenever $\xi_{k-1} \leq e^{-1}$ (otherwise the algorithm would have stopped), and operate on (17) to obtain

$$\xi_k e^{-\Delta} > (e^{-\ell_{k-1}} + \bar{\mu}) \frac{1 + e^{\ell_{k-1} - 0.5}\bar{\mu}}{1 + e^{\ell_{k-1}}\bar{\mu}} = e^{-\ell_{k-1}} + e^{-0.5}\bar{\mu}. \tag{18}$$

$$\text{Then,} \; -\ell_k = \lceil \ln \xi_k - \Delta \rceil = \lceil \ln(\xi_k e^{-\Delta}) \rceil > -\ell_{k-1}. \tag{19}$$

In addition, by definition of $\bar{\mu}$ and provided $\ell_{k-1} \leq n_0$ then

$$\xi_k e^{-\Delta} < \xi_k = e^{-\ell_{k-1}} + \bar{\mu} \leq e^{-\ell_{k-1}} + e^{-n_0+1} - e^{-n_0}$$
$$\leq e^{-\ell_{k-1}} + e^{-\ell_{k-1}+1} - e^{-\ell_{k-1}} = e^{-\ell_{k-1}+1}.$$

Application of ln to the latter inequality yields

$$\ln \xi_k - \Delta \leq -\ell_{k-1} + 1,$$

and since the right-hand side is integer valued, then also—

$$\ell_k = \lceil \ln \xi_k - \Delta \rceil \le -\ell_{k-1} + 1. \tag{20}$$

From (19) and (20), we reach

$$\ell_k = \ell_{k-1} - 1. \tag{21}$$

We have thus shown that if (Ri1) to (Ri3) are executed in Algorithm 1, so that $\mu_k = \bar{\mu}$, then the value ξ_k subsequently set at (R3) must satisfy $\lceil \ln \xi_k - \Delta \rceil = 1 - \ell_{k-1}$ and (21) will hold at (R1). As a consequence, the first iteration number k at which it happens that $\ln \xi_{k-1} - \Delta < -\ell_{k-1} - 0.5$ ($k \ge 2$ because this does not happen at $k = 1$), then it must be true that $\xi_{k-1}e^{-\Delta} > e^{-\ell_{k-2}} + e^{-0.5}\bar{\mu} = e^{-\ell_{k-1}-1} + e^{-0.5}\bar{\mu}$, as follows from (18) and (21). In this case the if condition in Algorithm 1 is not satisfied, (Re) is executed, and at (R3) it will happen that

$$\begin{aligned}
\xi_k &= (1+\sqrt{e})\xi_{k-1}e^{-\Delta} - e^{\lceil \ln \xi_{k-1} - \Delta \rceil - 0.5} \\
&= (1+e^{0.5})\xi_{k-1}e^{-\Delta} - e^{0.5}e^{-\ell_{k-1}-1} \\
&\ge (1+e^{0.5})\xi_{k-1}e^{-\Delta} + e^{0.5}(-\xi_{k-1}e^{-\Delta} + e^{-0.5}\bar{\mu}) \\
&= \xi_{k-1}e^{-\Delta} + \bar{\mu} \ge \xi_{k-1} + (1-e^{-1})\bar{\mu} > \xi_{k-1}.
\end{aligned} \tag{22}$$

Although in this case $\xi_k > \xi_{k-1}$, from (Re), (R3) and the definition of $\bar{h}$, then $\xi_k = \bar{h}(\xi_{k-1}e^{-\Delta}) \le e^{-\ell_{k-1}}$ and hence still $\ell_k = \ell_{k-1}$. As a consequence, $\xi_k e^{-\Delta} > \xi_{k-1}e^{-\Delta} > e^{-\ell_{k-1}-1} + e^{0.5}\bar{\mu} = e^{-\ell_k - 1} + e^{0.5}\bar{\mu}$. Therefore, the inequality $\xi_{k-1}e^{-\Delta} > e^{-\ell_{k-1}-1} + e^{-0.5}\bar{\mu}$ holds whenever $\ln \xi_{k-1} - \Delta < -\ell_{k-1} - 0.5$ and the above derivations show that $\xi_k > \xi_{k-1}$ whenever $\ln \xi_{k-1} - \Delta < -\ell_{k-1} - 0.5$. The sequence $\{\xi_k\}$ generated by Algorithm 1 is thus strictly increasing and $\xi_{k+1} - \xi_k \ge (1-e^{-1})\bar{\mu} > 0$, as follows from (16) and (22). We can thus bound the maximum number of iterations required as $F \le \left\lceil \dfrac{e^{-1} - e^{-n_0}}{(1-e^{-1})\bar{\mu}} \right\rceil$.

Since the sequence $\{\xi_k\}_{k=0}^F$ is strictly increasing, then the integer sequence $\{\ell_k\}_{k=0}^F$ is nonincreasing. Consider the sequence $\{k_i\}_{i=1}^{n_1}$. We have that $\mu_k \ne 0$ if and only if $k = k_i$ for some i, and, from the first part of the proof, that $k_1 = 1$, $\ell_{k_1} = n_0 - 1$ and $\ell_{k_i} = \ell_{k_i - 1} - 1$ for all $i = 1, \dots, n_1$. Since $\xi_{k_{n_1}-1} < e^{-1}$, we have that

$$-\ell_{k_{n_1}-1} - 0.5 + \Delta \le \ln \xi_{k_{n_1}-1} < -1,$$

and then $\ell_{k_{n_1}-1} \ge 1$. In consequence $\ell_{k_{n_1}} = \ell_{k_{n_1}-1} - 1 \ge 0$. Since $n_0 - 1 \ge \ell_{k_1} - \ell_{k_{n_1}} \ge n_1 - 1$, it follows that $n_1 \le n_0$.

Next, consider the quantities produced by Algorithm 1. Let $N \in \mathbb{N}$ be such that $\Delta_N = 2^{-(N+1)} < \Delta$ and $2^N > F$. Define the input u via $u(\tau_{N,k}) = \mu_k$ for $k = 1, \dots, F$ and $u(t) = 0$ otherwise. Note that $\|u\| \le \delta_2$. Consider the solution x corresponding to initial time s_N, initial condition δ_1 and input u. We have $x(\tau_{N,1}^-) = \delta_1 e^{-\Delta_N} \ge \delta_1 e^{-\Delta}$. Then

$$\begin{aligned}
x(\tau_{N,1}) &= h(\tau_{N,1}, \delta_1 e^{-\Delta_N}, u(\tau_{N,1})) \\
&\ge h(\tau_{N,1}, \delta_1 e^{-\Delta}, u(\tau_{N,1})) = \bar{h}(\delta_1 e^{-\Delta}) + \mu_1
\end{aligned}$$

where the last equality follows from the fact that $\sigma(\tau_{N,k}) = 0$ for all $\tau_{N,k}$ since $\tau_{N,k} \in [s_N, s_N + 1/2]$.

Then $x(\tau_{N,1}) \ge \bar{h}(\xi_0 e^{-\Delta}) + \mu_1 = \xi_1$. By induction, we can prove that $x(\tau_{N,i}) \ge \xi_i$ for $i = 1, 2, \dots, F$. Suppose that $x(\tau_{N,i}) \ge \xi_i$ for some i. This already holds for $i = 1$. Then, $x(\tau_{N,i+1}^-) = x(\tau_{N,i})e^{-\Delta_N} \ge x(\tau_{N,i})e^{-\Delta}$ and

$$\begin{aligned}
&x(\tau_{N,i+1}) = h(\tau_{N,i+1}, x(\tau_{N,i+1}^-), u(\tau_{N,i+1})) \\
&\ge h(\tau_{N,i+1}, \xi_i e^{-\Delta}, u(\tau_{N,i+1})) = \bar{h}(\xi_i e^{-\Delta}) + \mu_{i+1} = \xi_{i+1},
\end{aligned}$$

where we have used the properties and definition of h and the facts that $\xi_i \le 1$ and $\sigma(\tau_{N,i+1}) = 0$. As a consequence, it will happen that $x(\tau_{N,F}) \ge \xi_F \ge e^{-1}$, and the result is established with $x_0 = \delta_1$, $t_0 = s_N$ and $t = \tau_{N,F}$. ■

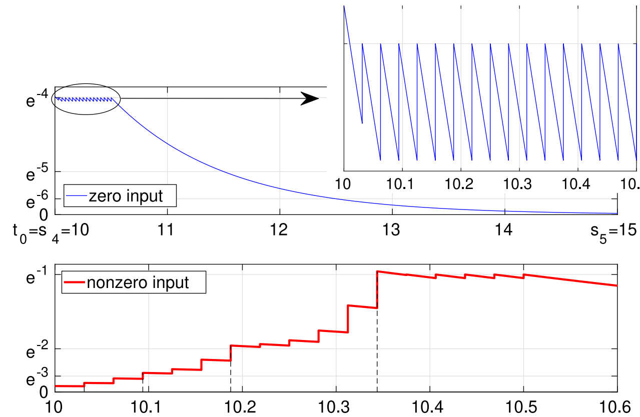

Fig. 3. Comparison between zero-input and nonzero-input trajectories with $t_0 = s_4 = 10$ and $x_0 = e^{-3.99}$. Top: zero-input trajectory. Bottom: trajectory corresponding to an input that takes the nonzero value 0.01 only at the impulse times $t_1 = 10 + 1/32$, $t_2 = 10 + 3/32$, $t_3 = 10 + 6/32$, and $t_4 = 10 + 11/32$.

Proof of Theorem 3.7. Suppose that Σ is iISS. Then there exist $\beta \in \mathcal{KL}$ and α, ρ_1 and $\rho_2 \in \mathcal{K}_\infty$ such that (6) holds. Pick any $\delta > 0$ so that $\beta(\delta, 0) + \delta < \alpha(e^{-1})$. Then, for all $t_0 \ge 0$, $|x_0| \le \delta$, $u \in \mathcal{U}$ such that $\|u\|_{\rho_1,\rho_2} \le \delta$ and $t \ge t_0$, if $x \in \mathcal{T}(t_0, x_0, u)$ then for all $t \ge t_0$

$$\alpha(|x(t)|) \le \beta(|x(t_0)|, t - t_0) + \|u\|_{\rho_1,\rho_2} < \alpha(e^{-1}).$$

Therefore $|x(t)| < e^{-1}$ for all $t \ge t_0$. Since the latter contradicts Lemma 3.8, it follows that Σ is not iISS. ■

We emphasize that given that $\gamma = \gamma^*$ as in Section 3.4 is the impulse–time sequence considered in Theorems 3.5, 3.6 and 3.7, and since $\sigma(t) = 0$ for every $t \in \gamma^*$, then we have actually shown that the given negative results hold for an impulsive system with time-invariant flow and jump maps.

Example 1. We next illustrate the main idea of Lemma 3.8. Consider the impulsive system (1) with (7)–(12) and $\gamma = \gamma^*$ as in Section 3.4. Consider the initial condition $x_0 = e^{-3.99}$ at initial time $t_0 = s_4 = 10$. The state evolution under zero input is shown in solid blue in the top plot of Fig. 3. First, note that $t_0 \in \gamma$ but the system does not jump at t_0. The system flows continuously until the next impulse–time $t_1 := t_0 + 1/32$. We have $e^{-4.5} < x(t_1^-) < e^{-4}$ and hence $x(t_1) = e^{-4}$. Subsequent flows and jumps will keep the state below but close to the value e^{-4} until after $t = 10.5$ when jumps will cease to occur frequently. From $t = 10.5$ to $t = s_5 = 15$, the system flows and hence the state decays exponentially towards 0. At $t = 15$, another blitz of jumps will occur (not shown in the figure), but now the state will be kept close to e^{-8} until $t = 15.5$. After this, again the state flows approaching 0 until the next blitz of impulses, and so on. This shows that the occurrence of impulses may delay but not prevent the convergence to 0.

By applying a very small but nonzero input, however, the state behavior may be drastically altered, as shown in red in the bottom plot of Fig. 3. At t_1, an input $u(t_1) = 0.01$ is applied, causing $e^{-4} < x(t_1) < e^{-3.5}$. Subsequent jumps will make the state closer and closer to e^{-3}, because the decay during flows is not sufficiently fast to counteract the impulse effects. At $t_2 := t_0 + 3/32$, a jump under zero input would have taken the state to the value e^{-3}; however, application of $u(t_2) = 0.01$ makes $e^{-3} < x(t_2) < e^{-2.5}$. Jumps continue to take the state farther and, by application of $u(t_3) = u(t_4) = 0.01$, with $t_3 = t_0 + 6/32$ and $t_4 = t_0 + 11/32$, we reach $x(t_4) > e^{-1}$. If instead of $t_0 = s_4$ we take $t_0 = s_5$, then we may make the state higher than e^{-1} using an input of smaller magnitude, because impulses occur more frequently between s_5 and $s_5 + 0.5$ than between s_4 and $s_4 + 0.5$. For any initial time, however, the maximum number of input applications will always be equal to $n_0 = -\lfloor \ln x_0 \rfloor = 4$. ∘

4. Discussion

The results obtained in Section 3 show that the 0-GUAS property as usually defined for impulsive systems is too weak for the system with inputs/disturbances to inherit any kind of meaningful stability with respect to small inputs. In fact, Theorems 3.5 and 3.6 show that the state of an impulsive system may be not small even when the initial condition and the inputs are small in magnitude. Lemma 3.8 shows that irrespective of the way in which the energy of an input is defined, the magnitude of a solution corresponding to an arbitrarily small initial condition and to an input with arbitrarily small energy may be not necessarily small.

One main reason for this lack of robustness is the fact that even if Zeno behavior is ruled out from admissible impulse–time sequences, i.e. impulse–time sequences cannot have finite limit points, impulses can occur arbitrarily frequently as time progresses. This is the case for the sequence γ^* defined in Section 3.4. When the initial time is not fixed, such as in the currently considered time-varying case, an appropriately large initial time t_0 can make impulses as frequent as desired even if the elapsed time $t - t_0$ is small. Although in practice it would be reasonable to assume that impulses cannot occur infinitely often, in some settings one cannot upper bound the number of impulses a priori in relation to elapsed time. Therefore, the 0-GUAS property as usually defined for impulsive systems is not very useful in the analysis/design of real world systems where the existence of modeling errors or disturbances is unavoidable, unless the number of impulses could be bounded in relation to elapsed time. Note that this type of bound on the number of impulses exists in the case of fixed dwell-time or average dwell-time sequences and, most generally, UIB sequences as per Haimovich and Mancilla-Aguilar (2018a).

These facts show the need for a stronger stability concept for impulsive systems. One way of strengthening stability is to mimic that considered in the theory of hybrid systems (Goebel et al., 2012) by taking into account, in the decaying term appearing in (4), the number of impulse–time instants contained in the interval $(t_0, t]$, namely $n_{(t_0,t]}$. This is achieved, for example, replacing the term $\beta(|x(t_0)|, t - t_0)$ by $\beta(|x(t_0)|, t - t_0 + n_{(t_0,t]})$ (see Mancilla-Aguilar & Haimovich, 2020). It can be easily shown that 0-GUAS in the usual sense implies this stronger 0-GUAS whenever the number of impulses that occur can be bounded in relation to elapsed time, a property that we referred to as the uniform incremental boundedness (UIB) of the impulse–time sequences (see Haimovich & Mancilla-Aguilar, 2018a; Mancilla-Aguilar & Haimovich, 2020).

In a forthcoming paper we will show that with this stronger definition of stability, 0-GUAS implies the CICS, the TS and the BEICS properties (Jayawardhana et al., 2010; Jayawardhana, 2010), where the latter is defined as follows: The system is bounded-energy-input converging-state (BEICS) if there exist $\rho_1, \rho_2 \in \mathcal{K}_\infty$ such that for all bounded $x \in \mathcal{T}(t_0, x_0, u)$, with $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$, and $u \in \mathcal{U}$ such that $\|u_{t_0}\|_{\rho_1,\rho_2} < \infty$, $x(t) \to 0$ as $t \to \infty$.

References

Angeli, D., Sontag, E. D., & Wang, Y. (2000). Further equivalences and semiglobal versions of integral input to state stability. *Dynamics and Control*, *10*(2), 127–149.

Barreira, L., & Valls, C. (2016). Robustness for stable impulsive equations via quadratic Lyapunov functions. *Milan Journal of Mathematics*, *84*, 63–89. http://dx.doi.org/10.1007/s00032-016-0251-8.

Briat, C., & Seuret, A. (2012). A looped-functional approach for robust stability analysis of linear impulsive systems. *Systems & Control Letters*, *61*, 980–988. http://dx.doi.org/10.1016/j.sysconle.2012.07.008.

Cai, C., & Teel, A. R. (2009). Characterizations of input-to-state stability for hybrid systems. *Systems & Control Letters*, *58*(1), 47–53.

Chen, W.-H., & Zheng, W. X. (2009a). Input-to-state stability and integral input-to-state stability of nonlinear impulsive systems with delays. *Automatica*, *45*, 1481–1488.

Chen, W.-H., & Zheng, W. X. (2009b). Robust stability and h_∞-control of uncertain impulsive systems with time-delay. *Automatica*, *45*, 109–117. http://dx.doi.org/10.1016/j.automatica.2008.05.020.

Clarke, F. H., Ledyaev, Y. S., & Stern, R. J. (1998). Asymptotic stability and smooth Lyapunov functions. *Journal of Differential Equations*, *149*(1), 69–114.

Dashkovskiy, S., & Feketa, P. (2017). Input-to-state stability of impulsive systems and their networks. *Nonlinear Analysis. Hybrid Systems*, *26*, 190–200.

Dashkovskiy, S., & Mironchenko, A. (2013). Input-to-state stability of nonlinear impulsive systems. *SIAM Journal on Control and Optimization*, *51*(3), 1962–1987.

Feketa, P., & Bajcinca, N. (2019). On robustness of impulsive stabilization. *Automatica*, *104*, 48–56. http://dx.doi.org/10.1016/j.automatica.2019.02.056.

Gao, L., & Wang, D. (2016). Input-to-state stability and integral input-to-state stability for impulsive switched systems with time-delay under asynchronous switching. *Nonlinear Analysis. Hybrid Systems*, *20*, 55–71.

Goebel, R., Sanfelice, R. G., & Teel, A. R. (2012). *Hybrid dynamical systems: Modeling, stability, and robustness*. Princeton University Press.

Hahn, W. (1967). *Stability of motion*. Springer-Verlag.

Haimovich, H., & Mancilla-Aguilar, J. L. 2018a. A characterization of iISS for time-varying impulsive systems. In *Argentine conference on automatic control (AADECA)*. Buenos Aires, Argentina..

Haimovich, H., & Mancilla-Aguilar, J. L. (2018b). A characterization of integral ISS for switched and time-varying systems. *IEEE Transactions on Automatic Control*, *63*(2), 578–585.

Haimovich, H., & Mancilla-Aguilar, J. L. (2020). Strong ISS implies strong iISS for time-varying impulsive systems. *Automatica*, Article 109224, available at http://arxiv.org/abs/1909.00858.

Haimovich, H., Mancilla-Aguilar, J. L., & Cardone, P. (2019). A characterization of strong iISS for time-varying impulsive systems. In *XVIII Reunión de trabajo en procesamiento de la información y control (RPIC)*. Bahía Blanca, Argentina. http://dx.doi.org/10.1109/RPIC.2019.8882162.

Hespanha, J. P., Liberzon, D., & Teel, A. (2008). Lyapunov conditions for input-to-state stability of impulsive systems. *Automatica*, *44*(11), 2735–2744.

Hong, S., & Zhang, Y. (2019). Input/output-to-state stability of impulsive switched delay systems. *International Journal of Robust and Nonlinear Control*, 1–22. http://dx.doi.org/10.1002/rnc.4705.

Jayawardhana, B. (2010). State convergence of dissipative nonlinear systems given bounded-energy input signal. *SIAM Journal on Control and Optimization*, *48*(6), 4078–4088.

Jayawardhana, B., Ryan, E., & Teel, A. (2010). Bounded-energy-input convergent-state property of dissipative nonlinear systems: an iISS approach. *IEEE Transactions on Automatic Control*, *55*(1), 159–164.

Lakshmikantham, V., Bainov, D., & Simeonov, P. S. (1989). Theory of impulsive differential equations. In *Modern Applied Mathematics*: *vol. 6*, World Scientific.

Lin, Y., Sontag, E. D., & Wang, Y. (1996). A smooth converse Lyapunov theorem for robust stability. *SIAM Journal on Control and Optimization*, *34*(1), 124–160.

Liu, B., Dou, C., & Hill, D. J. (2014). Robust exponential input-to-state stability of impulsive systems with an application in micro-grids. *Systems & Control Letters*, *65*, 64–73.

Liu, J., Liu, X., & Xie, W.-C. (2011). Input-to-state stability of impulsive and switching hybrid systems with time-delay. *Automatica*, *47*, 899–908.

Mancilla-Aguilar, J. L., & Haimovich, H. (2017). On zero-input stability inheritance for time-varying systems with decaying-to-zero input power. *Systems & Control Letters*, *104*, 31–37.

Mancilla-Aguilar, J. L., & Haimovich, H. (2020). Uniform input-to-state stability for switched and time-varying impulsive systems. *IEEE Transactions on Automatic Control*, http://dx.doi.org/10.1109/TAC.2020.2968580.

Ryan, E. P., & Sontag, E. D. (2006). Well-defined steady-state response does not imply CICS. *Systems & Control Letters*, *55*(9), 707–710.

Sontag, E. D. (1998). Comments on integral variants of ISS. *Systems & Control Letters*, *34*(1–2), 93–100.

Sontag, E. D. (2003). A remark on the converging-input converging-state property. *IEEE Transactions on Automatic Control*, *48*(2), 313–314.

Wang, Y.-E., Wang, R., & Zhao, J. (2013). Input-to-state stability of non-linear impulsive and switched delay systems. *IET Control Theory & Applications*, *7*(8), 1179–1185. http://dx.doi.org/10.1049/iet-cta.2012.0869.

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