The evolutionary minority game with local coordination

E. Burgos[a,*], Horacio Ceva[a], R.P.J. Perazzo[b]

[a] *Departamento de Física, Comisión Nacional de Energía Atómica, Avenida del Libertador 8250, Buenos Aires 1429, Argentina*

[b] *Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires and Instituto Tecnológico de Buenos Aires, Departamento de Investigación y Desarrollo, Avenida Eduardo Madero 399, Buenos Aires, Argentina*

Received 3 June 2003

Abstract

We discuss a modification of the evolutionary minority game (EMG) in which agents are placed in the nodes of a regular or a random graph. A neighborhood for each agent can thus be defined and a modification of the usual relaxation dynamics can be made in which each agent updates her decision depending upon her neighborhood. We report numerical results for the topologies of a ring, a torus and a random graph changing the size of the neighborhood. We find the surprising result that in the EMG a better coordination (a lower frustration) can be achieved if agents base their actions on local information disregarding the global trend in the self-segregation process.

PACS: 05.65.+b; 02.50.Le; 64.75.g; 87.23.Ge

Keywords: Evolutionary minority game; Local coordination

1. Introduction

There are a great number of situations in which a many agent system self-organizes by coordinating individual actions. Such coordination is usually achieved by agents with partial information about the system, and in some cases optimizing utility functions that conflict with each other. A similar situation is found in many particles, physical

* Corresponding author.

E-mail addresses: burgos@cnea.gov.ar (E. Burgos), ceva@cnea.gov.ar (H. Ceva), rperazzo@itba.edu.ar (R.P.J. Perazzo).

systems. The word "coordination" used in a social or economic context is then replaced by "ordering". Examples are the growth of a crystalline structure or a transition leading to some specific magnetic phase.

Interesting situations arise when the optimal configurations for different individuals do collide with each other. This can be due to the nature of the interactions between the particles as in a spin glass, or by boundary conditions, which prevent a global ordering or by the constitutive rules of a system of multiple players that prevent that all agents can win. In these cases it is said that the system displays some degree of *frustration*. An example of a frustrated multi agent system is given by the evolutionary minority game (EMG) [1] in which many players have to make a binary choice and the winning option is the one made by the minority. The similarities between some variants of the minority game and spin glasses have been discussed in great detail in Ref. [2].

A macroscopic signature of frustration is that the system can not accommodate into a single, optimal state in which the energy is a minimum, but it relaxes instead to one of many, suboptimal, equivalent configurations that are local minima in the energy landscape. In the random relaxation dynamic that is used for the EMG each player continually modifies her choice searching for a winning option. The final result is that the population is segregated into two parties that take opposite actions. This partition is not unique and also tends to reduce the frustration as much as possible by minimizing the number of losers.

The relaxation process is usually assimilated to the search of a solution of a combinatorial optimization problem in which it is possible a strategy of "divide and conquer" [3], i.e., circumstances in which one can attempt to divide the system into parts and search for separate optima in each part. Frustration arises when such local solutions can not be reassembled into a global optimum also fulfilling the boundary conditions.

A relevant example of the study of the global outcome of a local coordination strategy (i.e., involving only a fraction of the system) is Schelling's segregation model [4]. Agents of two kinds are placed in a square grid. The system relaxes to equilibrium allowing any two agents of different kinds to exchange places if they are surrounded by, say, a majority of agents of the opposite kind. In the present paper we discuss a relaxation dynamics for the EMG in which we impose a local coordination strategy. We borrow the picture of Schelling's models and place the players in a lattice. It is then possible to associate a neighborhood to each player, and thus implement a local coordination strategy letting each player to adjust her decision to the situation in her neighborhood. We call this model the local evolutionary minority game (LEMG).[1]

A previous work in this direction is Ref. [6] in which players are also located on the nodes of a grid, but are endowed with (two) strategies that are selected on the basis of their successful use. This work further imposes that both strategies have to be anticorrelated, i.e., they tend to produce opposite actions with the same input. We stress the *evolutionary* nature of the present model: players bear no memory of past

[1] Simultaneously with the submission of the present article, H.-J. Huan, B.-H. Wang, P.M. Hui and X.-S. Luo, have published a work along lines somewhat similar to the present article [5].

actions and do not have any strategy in the sense of Ref. [6],[2] to guide their actions. In spite of this difference, the effects of local coordination produce a similar ordered pattern. We pay special attention to the effects of such a local coordination in the optimization process.

2. The rules of LEMG

We first consider the traditional EMG. This involves N players that make one binary decision (0 or 1). Each player has a probability $p_i; i = 1, 2, \ldots, N$ of choosing, say, 0. Each player receives one point if her decision places her in the minority, and loses a point otherwise. When her account of points falls below 0, she changes $p_i \rightarrow p_i'$ with $p_i' \in [p_i - \Delta p, p_i + \Delta p]$, at random, and $\Delta p \ll 1$. Reflective boundary conditions are imposed at $p_i = 0, 1$. All agents are assumed to update the corresponding p_i's synchronically. It is customary to display the self-organization of the system through the probability density function $P(p)$ obtained in a statistical ensemble of systems that are allowed to relax to equilibrium. The value of $P(p)\,\mathrm{d}p$ is the fraction of the population having a probability between p and $p + \mathrm{d}p$ of choosing, say, 0. When the probabilistic relaxation is used, the asymptotic function $P(p)$ has a U-shape with two symmetric peaks at $p \simeq 0$ and 1 thus indicating that the N agents have segregated into two parties making opposite decisions. The relaxation process corresponds to the minimization of an "energy" function [7] given by the standard deviation σ defined by

$$\sigma^2 = \sum_A \mathscr{P}(A)(A - N/2)^2 , \tag{1}$$

where $\mathscr{P}(A)$ is the probability distribution of parties of A agents that have chosen 0. The value of σ^2 depends upon the properties of $P(p)$. In Ref. [7] it is proven that

$$\mathscr{E} \equiv \frac{\sigma^2}{N} = N(\bar{p} - 1/2)^2 + (\bar{p} - \overline{p^2}) , \tag{2}$$

where $\overline{p^s} = \int p^s P(p)\,\mathrm{d}p$. At equilibrium the linear dependence of $\mathscr{E}$ on N disappears, and σ^2 turns out to be an extensive magnitude proportional to N. A minimization of $\mathscr{E}$ is equivalent to find a distribution $P(p)$ with the smallest possible number of losers. In fact, σ^2 is related to the number of losers because

$$\sigma^2 = \langle (A - N/2)^2 \rangle = \frac{\langle (w - \ell)^2 \rangle}{4} = \frac{\langle (N - 2\ell)^2 \rangle}{4} , \tag{3}$$

where w (ℓ) is the number of winners (losers) and $\langle \cdots \rangle$ represents an ensemble average.

If one assumes naïvely $P(p) = \delta(p - \frac{1}{2})$ corresponding to a symmetric random walk (and thus eliminating the term $O(N)$ in Eq. (2)) one gets $\mathscr{E} = \frac{1}{4}$, while $P(p)$= constant yields $\mathscr{E} = \frac{1}{6}$. A better result is obtained with the usual random relaxation dynamics for the EMG. This yields [7] $\mathscr{E} \simeq \frac{1}{8}$. Energy and frustration remain linked to each other. For the EMG we can define frustration as $\mathscr{F} = \ell/N$; which fulfills $0 \leqslant \mathscr{F} \leqslant 1$.

[2] Two works where related ideas about local neighborhoods are developed are given in Ref. [6].

This definition may also be used for any system involving a game with multiple players. The value $\mathscr{F}=0$ corresponds to a situation such as the "majority game" in which a player is a winner if her decision is the same as the majority. This leads to situations that can be assimilated to a ferromagnetic phase (all the players (spins) have chosen the same option (orientation)). In the EMG there are less winners than losers, and therefore $\frac{1}{2} < \mathscr{F}_{EMG} \leqslant 1$. The lowest possible frustration for the EMG is reached when the N (odd) agents are coordinated to produce the largest possible minority, i.e., $(N-1)/2$. Thus the lowest possible frustration for a finite minority game is $\mathscr{F}^* = (1+1/N)/2$.

We now turn to the LEMG in which the ith player makes her decision depending upon the situation in her neighborhood $\mathscr{N}_i$. In order to define the neighborhoods we assume three possible spatial orderings. Two of them correspond, respectively, to a one-dimensional (1D) or a square two-dimensional (2D) regular array with periodic boundary conditions (i.e., respectively, a ring and a torus). In the third arrangement, the agents are placed in the nodes of a random undirected graph with a fixed number of neighbors for each agent so that a reciprocity relationship is automatically fulfilled (if node i is taken to be linked to node j, the reciprocal is also true). All neighborhoods are assumed to have the same (odd) number n of agents (we consider that $i \in \mathscr{N}_i$).

The rules of the LEMG are the same as for the EMG except for the important difference that an agent wins or loses points depending whether she is, or she is not, in the minority *of her own neighborhood*. No attention is paid to the agents that do not belong to $\mathscr{N}_i$. The LEMG coincides with the usual EMG when $\mathscr{N}_i$ coincides with the complete N-agent system. In the regular orderings the neighborhoods are respectively a segment or a square with an odd number of agents. The only agent that updates her p_i is located at the center of the square or segment. Notice that an agent may be in the minority (a winner) in her neighborhood and in the majority (a loser) when the entire system is considered, and vice versa.

Let us consider the simple example of an infinite linear chain of agents and a neighborhood with $n=3$. We define R_i to be the probability that the ith agent belongs to the minority of $\mathscr{N}_i$. We can thus write

$$R_i = (1-p_{i-1})p_i(1-p_{i+1}) + p_{i-1}(1-p_i)p_{i+1}\ , \tag{4}$$

where $-\infty \leqslant i \leqslant \infty$. The probability that all agents are winners is $R=\prod_i R_i$. Obviously, $R=1$ if and only if $R_i=1$, $\forall i$. This is possible only if $p_i=1$ and $p_{i\pm1}=0$. This corresponds to a pattern in which 0's and 1's alternate with a period of 2. Larger neighborhoods give also rise to periodic solutions with larger (even) periods. Any finite ring with an odd number of agents is frustrated because such periodicity can not fit along the chain.

Eq. (4) can be used to construct a (deterministic) relaxation dynamics to adjust the p_i's climbing along the gradient of R_i. We thus assume $p_i(t+1)=p_i(t)+\Delta p_i$ and set

$$\Delta p_i = \eta \partial R_i/\partial p_i = \eta(1-p_{i-1}-p_{i+1}) \tag{5}$$

with $1 \geqslant \eta > 0$. A stationary ($\Delta p_i = 0\ \forall i$) solution for this dynamics is $p_i=\frac{1}{2};\ \forall i$. This solution is unstable because any random perturbation of any p_i leads to a situation in which $\Delta p_i \neq 0\ \forall i$.[3] This dynamics stabilizes a pattern of 0's and 1's that alternate with each other. In fact if p_{i+1} and p_{i-1} are both greater (smaller) than $\frac{1}{2}$, then $\Delta p_i < 0 (> 0)$ thus forcing $p_i < \frac{1}{2} (> \frac{1}{2})$. This relaxation dynamics is therefore expected to lead to distributions $P(p)$ that vanish at $p = \frac{1}{2}$.

3. Results

3.1. Self-segregation

In Fig. 1 we show the results of $\mathscr{E} = \sigma^2/N$ as a function of the size parameter S defined as the ratio $S = n/N$ obtained in several numerical experiments. The value for $S = 1$ corresponds to $\mathscr{E}_{EMG}$ obtained for the EMG. In this section we only discuss results for $N = 121$. We have considered the topologies of a ring, a "square" torus (with $N = 11 \times 11$), and of random graphs. In all the cases considered, the values of $\mathscr{E}_S$ with $S \ll 1$ fulfill $\mathscr{E}_S < \mathscr{E}_{EMG}$. This feature is stressed in Fig. 1 with an horizontal line drawn at the value of $\mathscr{E}_{EMG}$. The value of $\mathscr{E}_S$ for regular 1D and 2D lattices grows with S and for $S \simeq 0.5$ becomes even larger that $\mathscr{E}_{EMG}$.

The corresponding density distributions $P(p)$ are shown in Fig. 2 for several values of S. These are compared with the distribution $P_{EMG}(p)$. We observe that $P_S(p)$ with $S \ll 1$ are always symmetric and U-shaped as $P_{EMG}(p)$ but they differ from this in the fact that they vanish around $p = \frac{1}{2}$. This agrees with the discussion given above for the linear chain. As we shortly discuss, this turns out to be a highly relevant and general feature of the LEMG. Such distributions are a better approximation to an ideal distribution

$$P(p) = \frac{N-1}{2N}\left[\delta(p) + \delta(p-1)\right] + \frac{1}{N}\delta(p - 1/2) \tag{6}$$

that yields the optimal value of $\mathscr{E}^*_{EMG} = 1/4N$ (and $\mathscr{F}^*_{EMG} = (1 + 1/N)/2$). A noticeable dip is produced for $S = (N-2)/N$. This can be understood in the following way. Assume that a symmetric distribution $P(p)$ has already developed and two agents are removed in order that the ith player can check her decision in her neighborhood of $N - 2$ agents. If the two agents that have been removed have $p > \frac{1}{2}$ ($p < \frac{1}{2}$) the ith agent has the single winning option of choosing $p_i \simeq 1$ ($p_i \simeq 0$). In the other cases (one player with $p > \frac{1}{2}$ and the other with $p < \frac{1}{2}$) her choices of approaching 0 or 1 have equal probability. The net result once all players have updated her respective p's in the same fashion is to force the resultant $P(p)$ to drop at $p = \frac{1}{2}$ and grow at $p = 0, 1$ as discussed for the linear chain. This argument can be extended for neighborhoods of other sizes.

3.2. The optimization problem

The shape of $P_S(p)$ changes with S. For intermediate values (for instance $S = \frac{81}{121}$) this distribution has radically changed from the shape of a U to a two wing profile

[3] There are other stationary solutions, such as a saw-tooth profile repeating the pattern $p_{i\pm1} = \frac{1}{2} \mp \varepsilon$ and $p_i = \frac{1}{2}$. These solutions are also unstable.

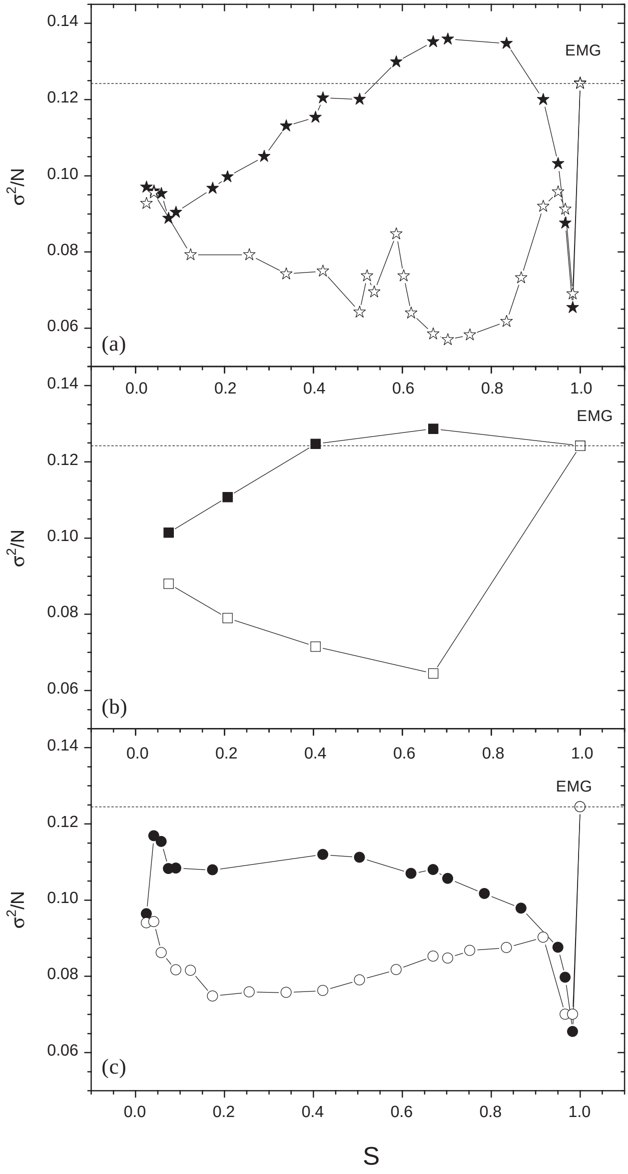

Fig. 1. σ^2/N as a function of the size parameter S for different topologies of the $N = 121$ players system. Panel (a) corresponds to a ring, panel (b) to a square torus and panel (c) to a random graph. Lines are drawn to guide the eye. Empty and filled symbols correspond respectively to results obtained with and without annealing (see the text). Data was obtained from 200 independent histories, of 5×10^5 time steps each, $\Delta p = 0.1$ and by averaging over the last 2000 time steps of all the histories. The annealing protocol consists in resetting all accounts to zero every 500 iterations. This is repeated 800 times. The fluctuations observed in the lower curve of panel (a) subsist in simulations with a much richer statistics (see the text).

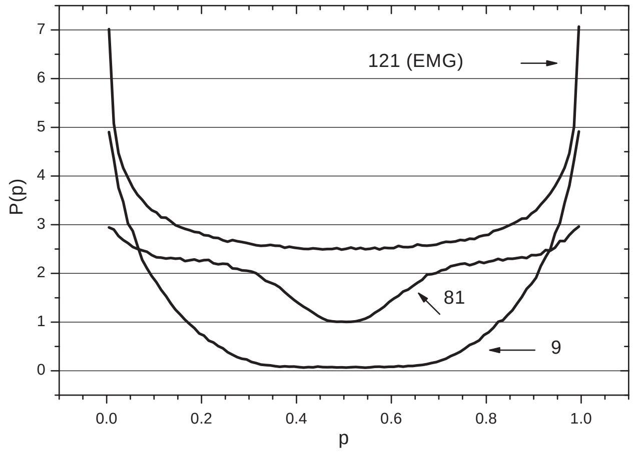

Fig. 2. Examples of the density distribution $P(p)$ for the linear chain and different size parameters ($S = \frac{9}{121}$, $\frac{81}{121}$ and 1 (EMG)). The second and third curves are offset by one and two units, respectively. Notice that the first two curves (almost) drop to zero around $p \simeq 0.5$.

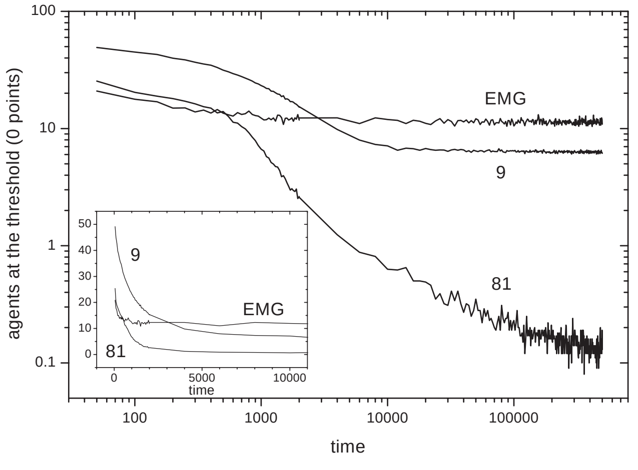

Fig. 3. Number of players with zero points as a function of the iteration number (time) for the same values of the size parameter as in Fig. 2. In the inset we display in a linear scale the same data up to 10,000 time steps, to put in evidence the presence of a fast and a slow dynamics.

with secondary maxima at both sides of $p = \frac{1}{2}$ still keeping the fact that $P_S(\frac{1}{2}) = 0$. This is associated to an increase in the number of "local winners". In fact after some time there are left almost no players that need to update their p_i's (see Fig. 3), while for $S \simeq 0$ or 1 the relaxation process reaches a dynamical equilibrium in which few players continuously update their p_i's.

For such intermediate values of S it is found that a minority is clearly defined in most neighborhoods and the corresponding agents are unambiguously induced to take one winning option. They therefore continue to accumulate points and cease to change their p_i's preventing the system to reach a more efficient self-segregation.

A situation like this has been extensively discussed in Ref. [8]. In these circumstances the relaxation process ceases to be effective to lower the energy and the system freezes in a configuration that is far from a better *local* optimum. Although the frozen microstates do depend upon initial conditions, the asymptotic density $P(p)$ is independent both because this is a density function that is associated with a macrostate of the system and because possible random variations are averaged out by repeating the relaxation process for a large ensemble of systems. Moreover, successive runs to obtain $\mathscr{E}$ changing the random initial configuration, yield essentially the same result. This is indeed to be expected because the energy is defined as the average of a statistical fluctuation, as given in Eq. (3).

The procedure to regain the true optimal self-segregation pattern, is to force the relaxation procedure by periodically removing the points that have been accumulated by every player, resetting their accounts to 0. This procedure changes *only* the situations in which the system is frozen but leaves unchanged situations in which this does not happen such as for instance for $S \simeq 0$ or 1.

These "annealing" episodes melt the system thus making it possible to reach the best local configurations. We have performed an annealed relaxation (with a fixed annealing protocol) for all three topologies. The results are displayed with open symbols in all three panels of Fig. 1. The fluctuations in the lower curve of panel a) for $S \simeq 0.6$ disappear for $N \geqslant 500$ thus indicating that it is a finite size artifact of the model.

A remarkable result displayed in Fig. 1 is that the composition of local optima always yields a better coordination than the one obtained within the framework of the EMG in which all agents are involved in the same relaxation process.

There are actually two situations to consider. One in which $S \simeq 0$ or 1, and the other where S is within these two extreme values. In the first case the LEMG always yields remarkably lower values of $\mathscr{E}$. This is indeed a general result, because holds true no matter the topology and the size of the system in which the players are located. In these cases no annealing is required because the system never gets quenched.

Outside the neighbourhood of $S \simeq 0$ or 1, the system gets quenched, and for regular topologies a poorer value of $\mathscr{E}$ (i.e., greater than $\mathscr{E}_{EMG}$) is obtained. However, for such values a better (lower) value of $\mathscr{E}$ can always be obtained if the annealing procedure is used to force the relaxation process beyond the frozen states. This actually means that a better coordination, i.e., a lower frustration, is achieved whenever the system is found in a configuration that corresponds to a composition of good local optima, and this is true in the whole range of values of S. If good local optima are guaranteed one finds values of $\mathscr{E}_S$ that are significantly lower that $\mathscr{E}_{EMG}$. A typical value of $\mathscr{E}_S \simeq \frac{1}{16}$ is obtained in this way that is half the value $\mathscr{E}_{EMG} \simeq \frac{1}{8}$.

Except for finite size effects, all the results presented in this subsection do not differ from those obtained for many other values of N that we have investigated.

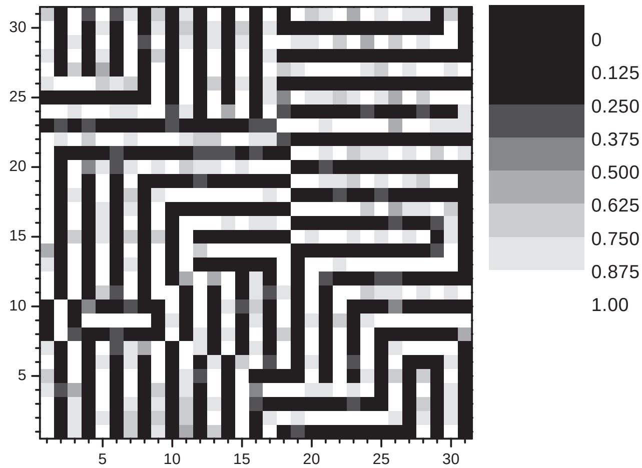

Fig. 4. An example of the LEMG for the topology of the torus. Domains in the map of probabilities for a 3×3 neighborhood. Each pixel represents a player; the corresponding p_i are shown as shades of gray.

3.3. The 2D case

A much richer situation is found for the case in which players are located on a grid with the topology of 2D torus. An example is shown in Fig. 4 in which $N = 31 \times 31$. The values of p are associated to shades of gray. Frustration can be perceived in the fact that there is not a single global ordering of black ($p \simeq 0$) and white ($p \simeq 1$) stripes for the whole array. These are instead grouped in domains with different orientations or with the same orientation but shifted with respect to each other. The relaxation process is fast in an initial stage and slows down once the domains have fully developed. The domain walls are a source of frustration. In fact, when such stage has been reached all the agents that have 0 points and continue to update their p_i's are located in the domain walls giving rise to a slow dynamics in which walls move enlarging or shrinking domains. The whole picture resembles a crystallization process; for a value of S greater than a critical threshold, all domains collapse into a single pattern of stripes. Frustration shows up as an indented (fuzzy) border in some of the stripes. These results will be discussed in detail elsewhere.

4. Conclusions

We have studied the organization pattern achieved by many agents playing an EMG with local coordination. We find important differences between the coordination achieved when the whole ensemble of agents participates in the same relaxation process or when local coordination is imposed.

According to the present results, the LEMG is an example in which a better coordination or, what is the same, a lower frustration, can be achieved provided that the self-organization is ruled by a local process, i.e., if agents govern their actions paying no attention to global trends of the system but rather to her immediate neighborhood. In the LEMG, this statement holds true even in the case in which very few agents are removed ($S \simeq (N-2)/N$) from the whole ensemble. The fact that a more efficient coordination is achieved by *ignoring* what happens to the total ensemble of players is expected to be a feature of a special kind of multi-agent systems. Other coordination problems may not behave in this fashion, thus allowing a classification of coordination games into classes linked to the type of optimization problem that is being "solved" by the ensemble of agents. Further investigation should be devoted to classify multi-agent games into those that fulfill this property and those that can not be optimized by braking them into pieces.

It is found that the LEMG displays some of the features that are typical of antiferromagnetic systems including the emergence of domains, frustration, fast and slow dynamics, etc., while also keeping the essence of multiagent models, as applied to social or economic organization. It therefore sheds light on the connection between those two bodies of knowledge.

E.B. has been partially supported by CONICET of Argentina, PICT-PMT0051.

References

[1] N.F. Johnson, P.M. Hui, R. Jonson, T.S. Lo, Phys. Rev. Lett. 82 (1999) 3360.
[2] D. Challet, M. Marsili, Phys. Rev. E 60 (1999) R6271;
D. Challet, M. Marsili, R. Zecchina, Phys. Rev. Lett. 84 (2000) 1824.
[3] D. Harel, Algorithmics. The Spirit of Computing, Addison Wesley, Reading, MA, 1987, p. 83.
[4] T.C. Schelling, J. Math. Sociol. 1 (1971) 143;
T.C. Schelling, Micromotives and Macrobehavior, W.W. Norton & Company, New York, 1978 (Chapter 4).
[5] H.-J. Huan, B.-H. Wang, P.M. Hui and X.-S. Luo, Physica A 321 (2003) 300.
[6] S. Moelbert, P. De los Rios, Physica A 303 (2002) 217;
M. Paczuski, K.E. Bassler, A. Corral, Phys. Rev. Lett. 84 (2000) 3185;
T. Kalinowski, H-J. Schulz, M. Briese, Physica A 277 (2000) 502.
[7] E. Burgos, H. Ceva, R.P.J. Perazzo, Phys. Rev. E 65 (2002) 036711.
[8] E. Burgos, H. Ceva, R.P.J. Perazzo, Physica A 294 (2001) 539.