Stochastic resonance and Brownian ratchets

A.J. Fendrik[a], L. Romanelli[b,*], R.P.J. Perazzo[c]

[a]*Departamento de Física J.J.Giambiagi, Facultad de Ciencias Exactas y Naturales. Universidad de Buenos Aires. (1428) Buenos Aires, Argentina*
[b]*Instituto de Ciencias, Universidad de General Sarmiento, J.M.Gutierrez 1150, (1613) Los Polvorines, Buenos Aires, Argentina*
[c]*Departamento de Investigación y Desarrollo, Instituto Tecnológico de Buenos Aires. Avda. Madero 399, (1106) Buenos Aires, Argentina*

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Abstract

We discuss the connections between Brownian ratchets (BR) and stochastic resonance (SR). We consider a periodic potential energy landscape with no left–right symmetry that is driven by an external force which can be derived from a potential that is periodic both in time and space. We show that this system presents two thermal enhancements within two different windows of the temperature. One is associated with a "coherent diffusion" by which particles jump back and forth between the minima of the periodic potential in synchrony with the external driving. The other is instead associated with a "coherent directional transport" by which particles hop synchronically from one minimum of the ratchet to the next. We calculate the current and the diffusion coefficients and show how transport undergoes a resonant enhancement. While the former is always present, the second only appears when left–right symmetry is broken.

Keywords: Transport phenomena; Diffusion; Brownian motors; Coherent transport; Thermal enhancement

*Corresponding author. Tel.: +54 11 4469 7534; fax: +54 11 4469 7506.
E-mail addresses: fendrik@df.uba.ar (A.J. Fendrik), lili@ungs.edu.ar (L. Romanelli), rperazzo@itba.edu.ar (R.P.J. Perazzo).

1. Introduction

Brownian Ratchets (BR) are devices out of equilibrium in which fluctuations make possible the directional transport of particles along a periodic potential with some left–right asymmetry. These devices that where first proposed by Smoluchowsky [1] and later discussed by Feynamnn [2] have deserved a great deal of attention in the literature (for reviews see Refs. [3,4]). There is a wide diversity of areas in which BRs are applied, for instance the working of molecular motors [5], the description of ion channels and molecular transport within cells [6] and the treatment of Parrondo's paradoxical games [7–9].

Stochastic resonance (SR) also represents a physical situation in which fluctuations play a similarly constructive role. It consists of a noise assisted enhancement by which power from the whole noise spectrum is pumped into a single mode that is coherent with an external driving force. This was first proposed in Ref. [10] to explain long term fluctuations in the Earth's climate but has later triggered a vast field of research [11] in which the biophysics of neural systems has a particularly important role [12–14]. In Ref. [15] the theory of SR has been discussed for two state systems and in Ref. [16], it is presented as a synchronization of the hopping mechanism between wells induced by the external periodic driving. In Refs. [17,18] a more general treatment of this theory is explained. A review of SR can be found in Ref. [19].

Although BR and SR are different physical phenomena, the fact that both can be placed within the same realm of noise-assisted non-equilibrium phenomena have induced several authors [6,20,21] to state that BR and SR may share some common underlying physical concepts. An attempt of a connection between SR and BR has been reported in Ref. [22] for a BR based on a Parrondo paradoxical game. The interference of two games plays the role of a random perturbation that gives rise to a resonant enhancement of the profit.

The present paper aims at bridging the gap between the theories of SR and BR by discussing how both phenomena may occur in the same physical system in spite of having deep physical differences. We do this by presenting a particular case of a BR in which the directional transport of particles can *resonantly* be enhanced within a window of values for the temperature. We refer to this device as a *stochastic resonant ratchet* (SRR). This model helps to discuss the similarities and the differences between the physics underlying BRs and SR. On the one hand the SRR can be assimilated to a spatially extended SR: particles are subject to thermal fluctuations and are placed in a periodic potential. A weak, periodic, external driving causes consecutive wells of the ratchet to alternate as absolute minima. Different levels of thermal noise are therefore expected to give rise to various regimes for the hopping of particles between consecutive minima, and thus to the transport of particles along the potential landscape. On the other hand, within the general theory of BR [4], the SRR can be framed as a *tilting ratchet* bearing also some similarity to the particular case of a *rocking ratchet* presented in Ref. [24], with a different type of external driving term.

The present work is organized as follows: in Section 2 we introduce the model under study. In Section 3 we discuss how this system presents two SRs at two different values of the amplitude of the external random fluctuations and in Section 4 we discuss the competition between directional transport and Brownian diffusion. Conclusions are drawn in Section 5.

2. The model

We consider overdamped particles placed in a periodic potential. The time evolution is given by the equation

$$\dot{x}(t) + \frac{\partial V_\alpha(x)}{\partial x} = F^{\,\mathrm{dr}}(x,t) + \xi(t)\ . \tag{1}$$

In Eq. (1), $x(t)$ represents the coordinate of the particle and $V(x)_\alpha$ is a one-dimensional, asymmetric periodic potential. We take

$$V_\alpha(x) = \begin{cases} -V_0(\cos[\frac{\pi}{\alpha}((\alpha+1)x+1)]+1) & \text{for } -1 \leqslant x \leqslant -\dfrac{1}{(\alpha+1)}\ , \\ V_0(\cos[\pi(\alpha+1)x]-1) & \text{for } -\dfrac{1}{(\alpha+1)} \leqslant x \leqslant 0 \end{cases} \tag{2}$$

fulfilling the periodicity condition $V_\alpha(x+1,t) = V_\alpha(x,t)$. Lengths along x are measured in units of the period of V_α and α ($\alpha > 0$) controls the left–right asymmetry. Solutions for $\alpha > 1$ in which the minimum in each well of the ratchet is displaced towards the right are equal to the time reversed solutions with $\alpha < 1$ in which minima are displaced towards the left. Particles are driven by the external periodic driving force $F^{\,\mathrm{dr}}(x,t)$. We take this to be the gradient of a time-dependent potential with a spatial periodicity that is twice the one of V_α in order that consecutive wells alternate in time as absolute minima. We therefore consider

$$F^{\,\mathrm{dr}}(x,t) = -\varepsilon\frac{\partial V^{\,\mathrm{dr}}(x,t)}{\partial x} = -\varepsilon\,\sin(\omega t)\frac{\mathrm{d}\,\sin(\pi x)}{\mathrm{d}x}\ . \tag{3}$$

Time is measured in units of the period $\tau = 2\pi/\omega$ of the external driving and ε is the coupling strength. All the calculations that we report were made for $V_0 = 5$, $\omega = .2$ and $\varepsilon = 12$. The last term in Eq. (1) is a Gaussian white noise fulfilling $\langle\xi(t)\xi(t')\rangle = 2k_BT\delta(t-t')$ thus representing a thermal bath of temperature T.

3. A double resonance

In the usual treatment of SR the response of the system is given by the amplitude of the induced noisy oscillations. We calculate this by integrating the stochastic Langevin equation (1). We also perform an average over several realizations (≈ 100) for randomly distributed initial conditions. The quality of such response is gauged through the signal-to-noise ratio (SNR) measured in its power spectrum. The present SRR allows to consider T as a control parameter and to look for the conditions of an SR. Accordingly, an enhancement of the SNR [15] should be expected for

$2k_BT_{SR} \simeq U$, where U is the height of the potential barrier between consecutive wells. In the present case U is directly related to the amplitude V_0 in Eq. (2). To cast the SRR into the above framework we consider that the output signal of the SRR is the coordinate $x(t)$ of the particles that are transported along the periodic potential. To evaluate the power spectrum we restrict $x(t)$ to the interval $[-1, 1]$ by defining $y(t) = x(t) - n$ for $n - 1 \leqslant x \leqslant n + 1$, and $n = 0, 2, \ldots$. The resulting power spectrum can be seen in Fig. 1 for several values of T. A clear peak can be observed at the same frequency of the external driving. There is also an irrelevant replica of this peak at a frequency three times larger that is a result of the truncation of the signal to the interval $[-1, 1]$. A second lower peak is found at twice the frequency of the external driving.

The second peak is a direct consequence of the broken left–right symmetry of the potential (for further discussion of this point see Refs. [17,23]). To see this we note that such potential asymmetry forces the particles located in consecutive wells to follow trajectories that approach the origin in one well while depart from it in the next one. The result is that the response of the system looks like a noisy, distorted square wave in which the broken symmetry manifests itself through the presence of even harmonics of the fundamental frequency (see Fig. 3). The situation is such that fluctuations can produce an optimal matching between both trajectories through an SR mechanism. The remarkable point is that the effect of a cascade of such matchings is to force particles to hop between consecutive wells in the direction prescribed by the broken symmetry of the potential. When $\alpha = 1$ particles no longer approach or depart the origin, even harmonics no longer appear and there is not a preferred direction for hopping.

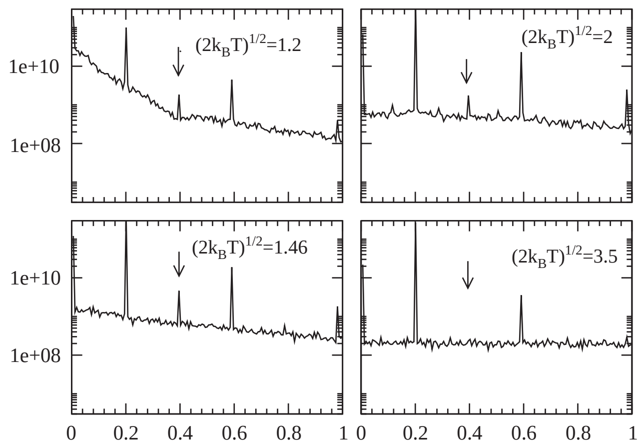

Fig. 1. Power spectrum of the response $y(t)$ for the values of T that are shown in each panel. The results correspond to an ensemble average of unfiltered signals originated in 50 initial conditions that have a Gaussian distribution restricted to the well at the left of the origin. The lowest peak occurs at a frequency equal to the external driving ($\omega = 0.2$). The next peak indicated by an arrow occurs at twice that frequency and undergoes a resonance at $(2k_BT_J)^{1/2} \simeq 1.46$. It cannot be detected at $T = T_{SR}$ (last panel at the right). See also Fig. 2.

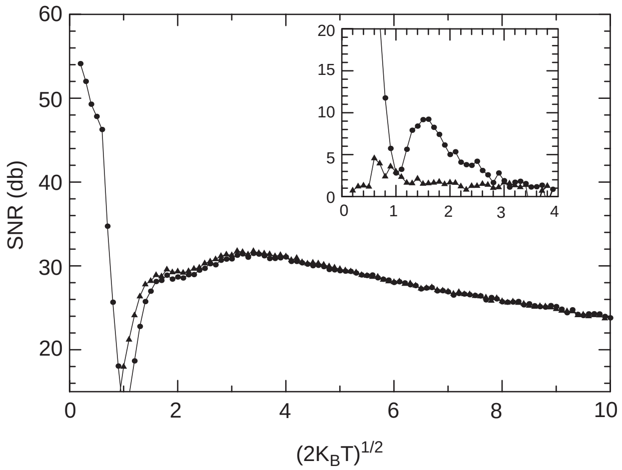

Fig. 2. Plot of the SNR associated with the two peaks ($\omega = 0.2$ and $\omega = 0.4$ commented in Fig. 1 as a function of $(2k_BT)^{1/2}$. The plot of the SNR of the second peak is displayed in the inset. In all cases we display the results for $\alpha = 3$ and $\alpha = 1$.

The T-dependence of the SNR of the two lowest peaks is shown in Fig. 2. Both display maxima at two different temperatures. The system therefore has *two separate* SRs. While the SNR of the first peak attains a maximum for a temperature T_{SR} as estimated from the general theory [15] of SR, the SNR of the next peak at twice the frequency that corresponds to the transfer of particles attains a maximum at a temperature T_J that is appreciably smaller than T_{SR}. The smaller fluctuations associated with T_J allow to detect the directionality of the potential. At $T = T_{SR}$, the larger fluctuations prevent instead the detection of its broken left–right symmetry. A higher T thus enhances the hopping between wells but on the other hand destroys the directional transport. This interpretation may be corroborated by comparing the T dependence of the SNR of both peaks as obtained with $\alpha = 1$ (symmetric potential) and $\alpha = 3$ (asymmetric potential). The SNR associated with the lowest frequency peak is left essentially unaltered by change of α while the highest frequency SR completely disappears when $\alpha = 1$.

4. Coherent hopping vs. Brownian diffusion

We now turn to study the SRR within the usual framework of BRs. To this end we have solved the thermal Langevin equation (1) through finite differences for a packet of particles corresponding to an ensemble of 1000 initial conditions that have a Gaussian distribution confined within the well at the left of the origin. In this case we do not restrict the motion to the interval $[-1, 1]$ and we instead calculate the average current (transport velocity) J along the (infinite) periodic potential. We also calculate

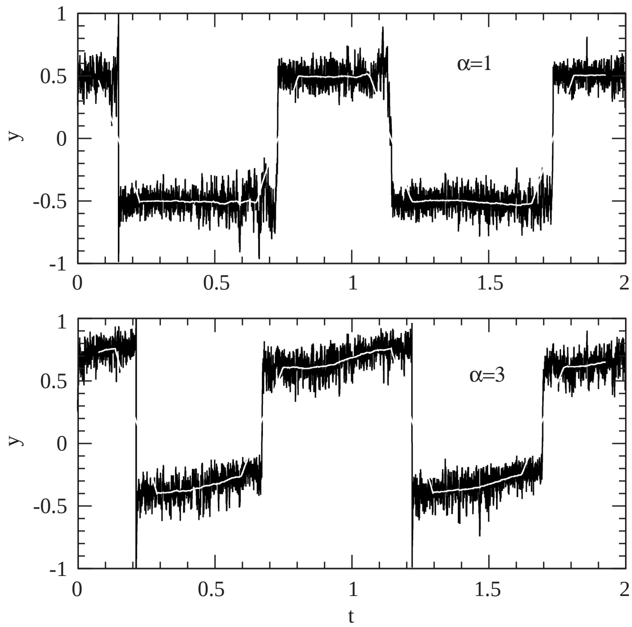

Fig. 3. Plot of $y(t)$ as a function of t for $T = T_J$ and $\alpha = 1$ and 3. The values of $y(t)$ in the asymmetric case ($\alpha = 3$) fluctuate around an average trend (shown as a white line) with a positive slope. The resulting distorted square wave has even and odd harmonics of the fundamental frequency. In the symmetric case ($\alpha = 1$) the average trend has a vanishing slope and even harmonics are no longer present. At a higher T, fluctuations wash out any effect of the broken left–right symmetry of V_α.

the effective diffusion coefficient D_{eff} as a function of T. These are defined as

$$J = \lim_{t\to\infty} J(t) = \lim_{t\to\infty} \left\langle \frac{x(t)}{t} \right\rangle , \tag{4}$$

$$D_{\mathrm{eff}} = \lim_{t\to\infty} D_{\mathrm{eff}}(t) = \lim_{t\to\infty} \left\langle \frac{1}{2t}(x^2(t) - \langle x(t)\rangle^2) \right\rangle . \tag{5}$$

In Eqs. (4) and (5) the symbol $\langle\cdot\rangle$ represents ensemble average. Except for very high values of T the results of $J(t)$ and $D_{\mathrm{eff}}(t)$ have a remarkably smooth behavior thus allowing to take the limits indicated Eqs. (4) and (5) in a straightforward way.

In Fig. 4 we show J and D_{eff} as a function of T. The plot of J is seen to display a maximum at a value $T_J < T_{SR}$ obtained above. The dependence of J on T shown in Fig. 4 is completely similar to the one shown in the inset of Fig. 2. Since J measures the transport of particles along the ratchet potential, this result agrees with the physical picture already presented by which the directional transport is resonantly enhanced by thermal fluctuations. The effect of different values of α on J can also be observed in Fig. 4. The two extreme situations of $T \ll T_{SR}, T_J$ and $T \gg T_{SR}, T_J$ can easily be understood. In the former case transport is impossible because very weak thermal fluctuations leave all particles confined in the initial well. For large T the

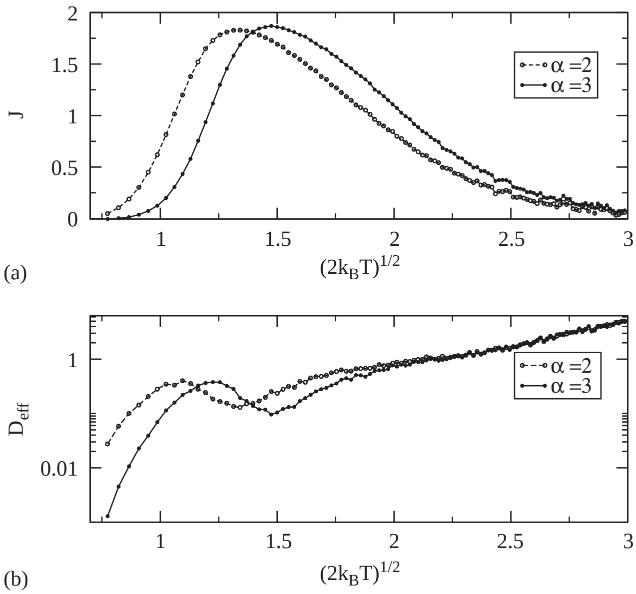

Fig. 4. Panels (a) and (b) show the current J and the effective diffusion coefficient D_{eff}, respectively, as a function of T. Note the log scale for D_{eff}. Compare the curves in panel (a) with the inset of Fig. 2.

current J drops because the asymmetry of the potential ratchet becomes irrelevant and particles diffuse in both directions.

The different transport regimes are displayed in Fig. 5 in which we plot the time evolution of the Gaussian packet for three different values of the temperature T. Successive rows starting from above display the packet after an interval equal to τ. Column (a) corresponds to a temperature $T < T_J$. It is seen that a directional transport prevails: after an interval τ always new peak appears in the next minimum of the ratchet potential. However, such transport is not optimal because there is always a sizable peak of lagging particles that have not yet hopped to the next well. In column (b) the temperature is $T = T_J$. The lagging particles have been drastically reduced and particles hop almost in perfect synchronism from one well to the next. Panel (c) corresponds to a temperature $T > T_J$. Directional transport has been lost and Brownian diffusion prevails. At such a high temperature particles are equally likely to hop in the forward or in the backward directions and after a time equal to 3τ the initial packet has been distributed almost evenly in four neighboring minima of the ratchet potential.

The competition between directional and Brownian diffusion is displayed in the plot of D_{eff} vs. T. As long as $T \leqslant T_J$ the particles of the Gaussian packet hop between consecutive wells in a preferred direction. At $T \simeq T_J$ a minimal distortion of the original packet is achieved due to an optimal coherence in hopping. This corresponds to a minimum of D_{eff}. For lower values of T such coherence is hindered because some particles are left behind. The Gaussian packet gets broader therefore giving rise to an increase of D_{eff}. For even lower values of T the confinement of

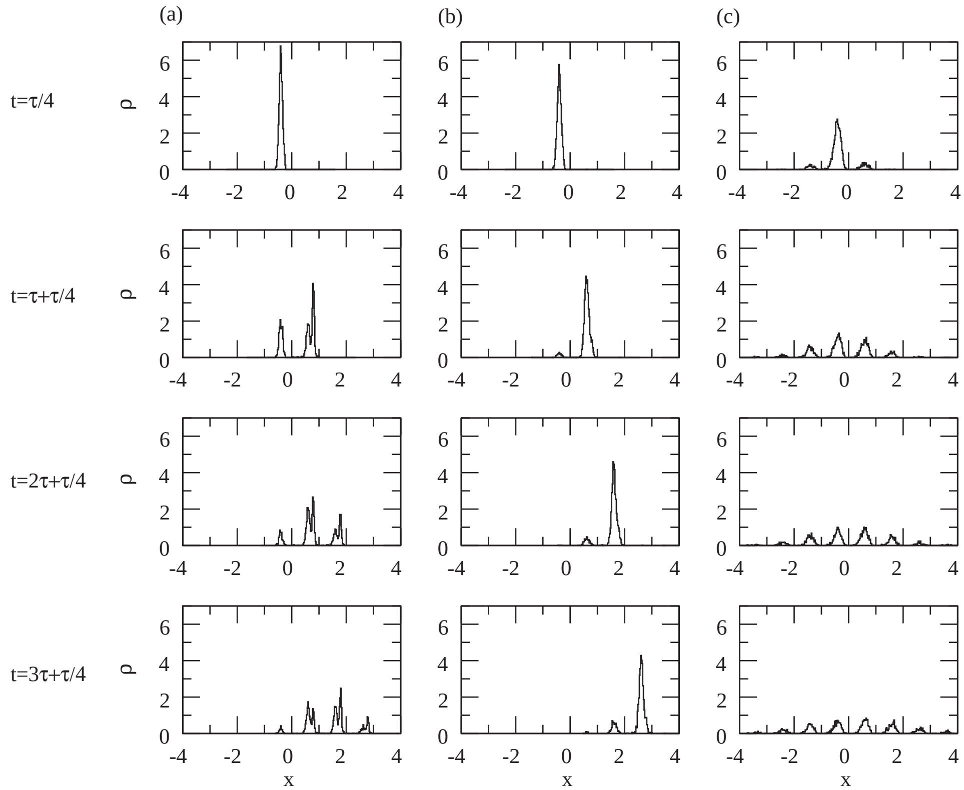

Fig. 5. Time evolution of an ensemble of particles that have initial conditions with a Gaussian distribution within the well placed at the left of the origin. Time runs downwards, the interval between neighboring rows is τ and the first row corresponds to a time $t = \tau/4$. Each column corresponds to a different temperature. Column (a) corresponds to $T < T_J$ ($(2k_BT)^{1/2} = 1.2$), column (b) to $T = T_J$ ($(2k_BT)^{1/2} = 1.46$) and column (c) to $T > T_J$ ($(2k_BT)^{1/2} = 2.5$). Consecutive minima are separated by a distance $x = 1$.

particles prevents any diffusion and $D_{\mathrm{eff}} \to 0$. For $T > T_J$, a different regime starts to prevail. The Gaussian packet gets distorted because particles increasingly hop in the backward direction. When $T \gg T_J$ there is no trace left of the original Gaussian packet and D_{eff} grows monotonically.

5. Conclusions

We have presented a model consisting in a periodically driven BR that exhibits a double resonance. One is associated with a broken symmetry of the potential and shows up as a magnification of the directional current of particles within a limited window of the temperature. This enhancement is achieved due to an optimal coherence in the hopping of particles between neighboring wells as checked for a

(Gaussian) packet of particles initially concentrated in one minimum of the ratchet potential. This resonance takes place at temperature T_J that is significantly lower than the one estimated from the theory of SR. The second enhancement appears at a temperature $T = T_{SR} > T_J$ and is properly described by the well-known general theory of SR. This turns out to be insensitive to the broken left–right symmetry of the ratchet potential and corresponds to an increase in the effective diffusion coefficient. By the time in which the system undergoes this SR, directional transport is hindered, coherence in the hopping between wells is impaired and particles hop between wells *regardless* of the directionality of the ratchet potential.

This discussion indicates that both enhancements, in spite of being controlled by external stochastic fluctuations, have distinct physical origins. The enhancement at a lower temperature is associated with the maximum possible difference between the escape rates in the forward and backward directions and it only appears in periodic potentials that lack left–right symmetry. The enhancement at a higher temperature is instead the only one that truly corresponds to the well-known SR theory. It appears at a T corresponding to purely diffusive regime and can be found also in symmetric periodic potentials, and corresponds to particles jumping back and forth in synchronization with the external driving.

It has been argued in Ref. [28] that the output signal $x(t)$ on a periodic substrate does not show SR. In the present model the SR is seen not for the output signal $x(t)$ but for $y(t) = x(t) - n$, for $n - 1 \leqslant x \leqslant n + 1$ and $n = 0, 2, \ldots$; i.e., for the restriction of $x(t)$ to the interval $[-1, 1]$. The occurrence of SR in $y(t)$ at a frequency ω that we report in this paper should, however, not be considered a numerical artifact. The filtering of $x(t)$ is meant to eliminate zero-frequency components of $x(t)$ (that would make this output signal unbound) and corresponds to the synchronization reported in Ref. [16] between the directionless hopping among neighboring wells with the external periodic driving.

The present discussions of SR and BR apply to extended systems. It is possible therefore to claim that the present family of SRRs can also be framed within the same realm of systems that undergo spatiotemporal SR [25–27] in which an extended regular pattern synchronizes with an external periodic driving at a sharp, well-defined noise level. This also suggests that this phenomenon can also be found in BR involving periodic potentials with a higher dimension.

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