ISS implies iISS even for switched and time-varying systems (if you are careful enough)☆

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ABSTRACT

For time-invariant systems, the property of input-to-state stability (ISS) is known to be strictly stronger than integral-ISS (iISS). Known proofs of the fact that ISS implies iISS employ Lyapunov characterizations of both properties. For time-varying and switched systems, such Lyapunov characterizations may not exist, and hence establishing the exact relationship between ISS and iISS remained an open problem, until now. In this paper, we solve this problem by providing a direct proof, i.e. without requiring Lyapunov characterizations, of the fact that ISS implies iISS, in a very general time-varying and switched-system context. In addition, we show how to construct suitable iISS gains based on the comparison functions that characterize the ISS property, and on bounds on the function f defining the system dynamics. When particularized to time-invariant systems, our assumptions are even weaker than existing ones. Another contribution is to show that for time-varying systems, local Lipschitz continuity of f in all variables is not sufficient to guarantee that ISS implies iISS. We illustrate application of our results on an example that does not admit an iISS-Lyapunov function.

1. Introduction

Both input-to-state stability (ISS) and integral-input-to-state stability (iISS) can be considered nonlinear-system extensions of the type of stability that a linear time-invariant system with inputs is known to have. The norm of the state of a system that is either ISS or iISS can be bounded by the sum of a term depending only on the initial state norm and decaying asymptotically to zero, and a term depending only on the input. Loosely speaking, for ISS the input-dependent term depends on a bound on the maximum input amplitude whereas for iISS, the dependence is on the input energy.

The concepts of ISS and iISS, originally introduced for time-invariant continuous-time systems in, respectively, Sontag (1989, 1998), were subsequently extended and studied for other classes of systems: time-varying systems (Edwards, Lin, & Wang, 2000), discrete-time systems (Jiang & Wang, 2001), switched systems (Haimovich & Mancilla-Aguilar, 2018; Mancilla-Aguilar & García, 2001), impulsive systems (Hespanha, Liberzon, & Teel, 2008), hybrid systems (Cai & Teel, 2009; Noroozi, Khayatian, & Geiselhart, 2017) and infinite dimensional systems (Dashkovskiy & Mironchenko, 2013; Mironchenko & Wirth, 2018).

For time-invariant systems, ISS was shown to be equivalent to the existence of a dissipation-form ISS-Lyapunov function (Sontag & Wang, 1995), and analogously for iISS (Angeli, Sontag, & Wang, 2000a). Based on these Lyapunov characterizations of ISS and iISS, it is easy to see that if a system is ISS, then it also is iISS (Angeli et al., 2000a). In other words, for time-invariant systems, it is known that ISS implies iISS. The only requirement for the latter implication to hold is that the function f defining the system dynamics $\dot{x} = f_{ti}(x, u)$ be locally Lipschitz. Loosely speaking, classes of systems where ISS is equivalent to the existence of a dissipation-form ISS-Lyapunov function make the implication ISS $\Rightarrow$ iISS hold. For such classes of systems, the latter implication can be established almost identically as for continuous-time time-invariant systems. Some of these classes are, for example, time-invariant switched systems under arbitrary switching (Mancilla-Aguilar & García, 2001) and time-invariant hybrid systems (Cai & Teel, 2009; Noroozi et al., 2017).

Although many other characterizations of ISS (Liberzon & Shim, 2015; Sontag & Wang, 1995, 1996) and iISS (Angeli et al., 2000a; Angeli, Sontag, & Wang, 2000b; Haimovich & Mancilla-Aguilar, 2018) exist, until now the only known way of proving that ISS implies iISS was based on a dissipation-form ISS-Lyapunov function as mentioned above. Therefore, classes of systems for which ISS is not necessarily equivalent to the existence of a

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dissipation-form ISS-Lyapunov function, such as continuous-time time-varying systems (see Edwards et al., 2000) or switched systems under restricted switching (see Haimovich & Mancilla-Aguilar, 2018), cause the question of whether ISS implies iISS to remain open.

In Kellett, Wirth, and Dower (2013), the concepts of ISS and iISS with respect to two measures were analyzed for continuous-time time-invariant systems. The main difference with respect to the standard corresponding properties is that the set to which the state asymptotically converges need not be compact. In Kellett et al. (2013), it is also shown that the noncompactness of the convergence sets may destroy the link between ISS and iISS, in the sense that a system may be ISS but not iISS, both with respect to two measures. Considering time as a state variable and under some regularity conditions, an ISS continuous-time time-varying system can be written as a time-invariant system which is ISS with respect to an unbounded invariant set. Therefore, the results in Kellett et al. (2013) suggest that the implication ISS $\Rightarrow$ iISS may no longer hold true for general time-varying systems.

In this paper, and as the title claims, we show that ISS implies iISS even for time-varying and switched systems, and in cases where Lyapunov characterizations of ISS and iISS may not exist. We thus contribute to solving a control theory problem that still remained open. We show that the implication holds under specific boundedness and continuity conditions on the function defining system dynamics. These boundedness and continuity conditions are weaker than local Lipschitz continuity of $f_{\rm ti}$ when particularized to time-invariant systems $\dot{x} = f_{\rm ti}(x, u)$, because Lipschitz continuity with respect to the input variable u is not required. However, and as a second contribution, we provide an example of a time-varying system $\dot{x} = f_{\rm tv}(t, x, u)$ with $f_{\rm tv}$ locally Lipschitz that is ISS but not iISS. This shows that local Lipschitz continuity in all variables is not sufficient to guarantee that ISS implies iISS and thus one must be careful when claiming that ISS implies iISS. A third contribution is to show how suitable iISS gains may be constructed based on bounds on the system function and on the comparison functions characterizing ISS.

The remainder of the paper is organized as follows. Section 2 begins with a brief description of the notation employed, describes the type of systems considered, defines the required stability properties (ISS and iISS), explains the known relationship between these properties, and ends with an example of a time-varying system that is ISS but not iISS. Our main results are contained in Section 3, where we show that ISS implies iISS under specific conditions. Section 4 provides an example of a switched system that is ISS and hence is shown to be iISS by means of the current results. Conclusions are provided in Section 5. The Appendices contain the proofs of some intermediate technical results.

2. Preliminaries

2.1. Notation

The reals, nonnegative reals, positive reals, naturals and nonnegative integers are denoted by $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$, $\mathbb{N}$ and $\mathbb{N}_0$, respectively. $|\xi|$ denotes the Euclidean norm of any $\xi \in \mathbb{R}^k$. For any $k \in \mathbb{N}$ and any $r \geq 0$, we define $B_r^k := \{\xi \in \mathbb{R}^k : |\xi| \leq r\}$. We write $\sigma \in \mathcal{K}$ if $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous, strictly increasing, and $\sigma(0) = 0$. We write $\sigma \in \mathcal{K}_\infty$ if $\sigma \in \mathcal{K}$ and σ is unbounded. Let $\mathcal{M}(\mathbb{R}^m)$ be the set of Lebesgue measurable functions $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$. The elements of $\mathcal{M}(\mathbb{R}^m)$ will be referred to as inputs. For any input u, $\|u\| = \text{ess sup}\{|u(t)| : t \geq 0\}$. $L^\infty_{\rm loc}(\mathbb{R}^m)$ denotes the set of all the locally essentially bounded inputs. Given $\chi \in \mathcal{K}$, $L^1_{{\rm loc},\chi}(\mathbb{R}^m)$ is the set of all the inputs u such that $\chi(|u|)$ is locally integrable. For $\chi \in \mathcal{K}$ and $u \in \mathcal{M}(\mathbb{R}^m)$, we define $\|u\|_\chi = \int_0^\infty \chi(|u(t)|)dt$. For any input u and interval $I \subset \mathbb{R}_{\geq 0}$, $u_I : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is the input defined by $u_I(s) = u(s)$ if $s \in I$ and $u_I(s) = 0$ otherwise. For $a, b \in \mathbb{R}$, we define $a \wedge b := \min\{a, b\}$.

2.2. Systems considered

This work deals with a parametrized family of time-varying control systems of the general form

$$\dot{x}(t) = f_\lambda(t, x(t), u(t)), \quad \lambda \in \Lambda, \tag{1}$$

with Λ an arbitrary nonempty set – the parameter set, – and where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ for $t \in \mathbb{R}_{\geq 0}$, and for every $\lambda \in \Lambda$, $f_\lambda : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ satisfies $f_\lambda(t, 0, 0) = 0$ for all $t \geq 0$ (more precise assumptions on f_λ will be given in Section 3.1).

Remark 2.1. When we speak about a parametrized family of time-varying systems, we just mean that to each $\lambda \in \Lambda$ we have associated a system with inputs whose system function is denoted by f_λ. The set Λ can be any nonempty set without any additional structure. This is just a way to describe a family of systems, and the parameters λ play the role of labels identifying each of the systems in the family. ∘

For an input u and $\lambda \in \Lambda$, a (forward) solution of (1) is a locally absolutely continuous function $x : [t_0, t_f) \to \mathbb{R}^n$, with $0 \leq t_0 < t_f$, such that $\dot{x}(t) = f_\lambda(t, x(t), u(t))$ for almost all $t \in [t_0, t_f)$. A solution is maximally defined (in forward time) if there does not exist another solution $\tilde{x} : [t_0, \tilde{t}_f)$ of (1) such that $\tilde{t}_f > t_f$ and $x(t) = \tilde{x}(t)$ for all $t \in [t_0, t_f)$.

Definition 2.2. We say that $\mathcal{U} \subset \mathcal{M}(\mathbb{R}^m)$ is a set of admissible inputs if:

(a) for any interval $I \subset \mathbb{R}_{\geq 0}$ and any[1] $u \in \mathcal{U}$, $u_I \in \mathcal{U}$;
(b) for each $u \in \mathcal{U}$, $t_0 \geq 0$, $\xi \in \mathbb{R}^n$ and $\lambda \in \Lambda$, there exist a unique maximally defined solution x of (1) which satisfies $x(t_0) = \xi$;
(c) $L^\infty_{\rm loc}(\mathbb{R}^m) \subset \mathcal{U}$, i.e. locally essentially bounded Lebesgue measurable functions are always admissible as inputs.

For a given set $\mathcal{U}$ of admissible inputs, we will denote by $x(\cdot, t_0, \xi, u, \lambda)$ the unique maximally defined solution of (1) corresponding to $u \in \mathcal{U}$ and $\lambda \in \Lambda$ which satisfies $x(t_0, t_0, \xi, u, \lambda) = \xi$ and by $I_{t_0,\xi,u,\lambda}$ its forward interval of definition. We say that $x(\cdot, t_0, \xi, u, \lambda)$ is forward complete if $I_{t_0,\xi,u,\lambda} = [t_0, \infty)$, and that the family of systems is forward complete if $x(\cdot, t_0, \xi, u, \lambda)$ is forward complete for every $\lambda \in \Lambda$, $t_0 \geq 0$, $\xi \in \mathbb{R}^n$ and $u \in \mathcal{U}$.

It is clear that the setting (1) is sufficiently general so as to describe time-invariant and time-varying, single systems as well as families of systems. For clarity and future reference, we provide the following definitions.

Definition 2.3. A family of the form (1) is said to be

(a) time-invariant if for every $\lambda \in \Lambda$, f_λ does not depend on $t \in \mathbb{R}_{\geq 0}$.
(b) a (single) system if Λ has a single element.
(c) a time-invariant system if the family is both time-invariant and a single system.

The setting (1) can also be used to appropriately describe switched systems. This is explained in the next subsection.

2.3. Switched systems as parametrized families of systems

A switched system can be defined by an equation of the form

$$\dot{x}(t) = f_{\rm sw}(t, x(t), u(t), \sigma(t)), \tag{2}$$

[1] Recall that $u_I(s) = u(s)$ if $s \in I$ and $u_I(s) = 0$ otherwise.

where $f_{sw} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \times \Gamma \to \mathbb{R}^n$, with Γ a nonempty set – the index set –, and $\sigma : \mathbb{R}_{\geq 0} \to \Gamma$ a piecewise constant[2] and right-continuous function – the switching signal –. The symbol σ is employed to denote the whole function whereas $\sigma(t)$ is employed to denote the value of σ at time t [i.e. $\sigma(t)$ indicates the active subsystem at time t]. Therefore, σ and $\sigma(t)$ represent different objects and belong to different sets: σ belongs to the set $\mathcal{S}_{all}$ of all the possible switching signals and $\sigma(t) \in \Gamma$. Each switching signal σ gives rise to a time-varying system with inputs, namely $\dot{x}(t) = f_\sigma(t, x(t), u(t))$ (here σ is interpreted as a label) whose system function $f_\sigma : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is defined by

$$f_\sigma(t, \xi, \mu) := f_{sw}(t, \xi, \mu, \sigma(t)). \quad (3)$$

In the context of switched systems, if every switching signal is admissible, then we speak about "arbitrary switching"; if switching is constrained to have some minimum dwell-time, we speak about "dwell-time switching". In every case, we may define a set $\mathcal{S} \subset \mathcal{S}_{all}$ containing the admissible switching signals (e.g. all the switching signals under arbitrary switching and only those that have a minimum dwell time under dwell-time switching). By setting $\Lambda = \mathcal{S}$ in (1), it is easy to see that a switched system can be identified with a parametrized family of time-varying systems, where the admissible switching signals play the role of parameters (i.e. labels). Note that although the index set Γ may be finite, the parameter set Λ is usually infinite since the latter set equals $\mathcal{S}$ and not Γ. This is illustrated in the following example.

Example 1. Consider the switched linear system with two modes and under arbitrary switching, given by

$$\dot{x}(t) = f_{sw}(t, x(t), u(t), \sigma(t)) := A_{\sigma(t)}x(t) + b_{\sigma(t)}u(t),$$

with $A_i \in \mathbb{R}^{n\times n}$ and $b_i \in \mathbb{R}^n$ for $i \in \Gamma := \{1, 2\}$. The admissible switching signals $\mathcal{S}$ are all the piecewise constant and right-continuous functions $\sigma : \mathbb{R}_{\geq 0} \to \Gamma$. For each switching signal $\sigma \in \mathcal{S}$, the corresponding time-varying system is $\dot{x}(t) = f_\sigma(t, x(t), u(t))$, with $f_\sigma(t, \xi, \mu) \equiv A_{\sigma(t)}\xi + b_{\sigma(t)}\mu$. Note that there exist an infinite number of admissible switching signals, and hence $\Lambda = \mathcal{S}$ is infinite. ◦

2.4. Stability properties: ISS and iISS

For the parametrized family of systems (1) and the set of admissible inputs $\mathcal{U}$ we will consider the following input-to-state stability properties which are uniform over the set of parameters Λ. Their definitions are very natural extensions of those introduced in Sontag (1989, 1998) for time-invariant systems.

Definition 2.4. Consider a family of systems (1) and a set $\mathcal{U}$ of admissible inputs.

(a) The family of systems (1) is *input-to-state stable* (ISS) if there exist $\beta \in \mathcal{KL}$ and $\rho \in \mathcal{K}_\infty$ such that for every $t_0 \geq 0$, $\xi \in \mathbb{R}^n$, $u \in \mathcal{U}$ and $\lambda \in \Lambda$, $x(\cdot) = x(\cdot, t_0, \xi, u, \lambda)$ satisfies the following estimate for all $t \in I_{t_0,\xi,u,\lambda}$:

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \rho(\|u\|). \quad (4)$$

The function ρ will be referred to as an ISS gain.

(b) The family of systems (1) is *integral input-to-state stable* (iISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that for every $t_0 \geq 0$, $\xi \in \mathbb{R}^n$, $u \in \mathcal{U}$ and $\lambda \in \Lambda$, $x(\cdot) = x(\cdot, t_0, \xi, u, \lambda)$ satisfies the following estimate for all[3] $t \in I_{t_0,\xi,u,\lambda}$,

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma_1\left(\|u\|_{\gamma_2}\right). \quad (5)$$

The function γ_2 will be referred to as an iISS gain.

Remark 2.5. Due to causality and the Markov property, equivalent definitions of ISS and iISS are obtained if u is replaced by $u_{[t_0,t]}$ in (4) and (5), respectively. Note that we do not require the solutions of (1) to be defined for all $t \geq t_0$ in the definitions of ISS and iISS. Notwithstanding, well-known results for ordinary differential equations show that if the family (1) is ISS (resp. iISS) then for every input $u \in \mathcal{U}$ such that $\|u_{[t_0,t]}\| < \infty$ (resp. $\|u_{[t_0,t]}\|_{\gamma_2} < \infty$), it happens that $[t_0, t] \subset I_{t_0,\xi,u,\lambda}$. Therefore, the solutions of (1) are forward complete for all $u \in L^\infty_{loc}(\mathbb{R}^m)$ (resp. $u \in L^1_{loc,\gamma_2}(\mathbb{R}^m) \cap \mathcal{U}$), when the family of systems is ISS (resp. iISS with iISS gain γ_2). ◦

Remark 2.6. When a family of systems is ISS for some set of admissible inputs $\mathcal{U}$, then it also is ISS for any other set of admissible inputs $\mathcal{U}'$ and, in particular, for the set $L^\infty_{loc}(\mathbb{R}^m)$, which is the set of admissible inputs usually considered in the ISS framework. As a consequence, we do not need to make the set $\mathcal{U}$ explicit when dealing with the ISS property. By contrast, the set of admissible inputs $\mathcal{U}$ plays a relevant role in the case of the iISS property, since, for example, a family of systems could be iISS for $\mathcal{U} = L^\infty_{loc}(\mathbb{R}^m)$ but not for a larger $\mathcal{U}$. Considering sets $\mathcal{U}$ larger than $L^\infty_{loc}(\mathbb{R}^m)$ in the iISS property allows us to study the dependence of the states upon inputs which are not locally essentially bounded but have "finite energy" on every finite interval. ◦

2.5. Known relationship between ISS and iISS

A natural question is: what is the precise relationship between the ISS and iISS properties? Since $\|u\|$ and $\|u\|_{\gamma_2}$ are nonequivalent ways of measuring the size of an input u, this question cannot be answered directly from the very definitions of the properties. By means of Lyapunov characterizations, this question has been answered for time-invariant systems, when (1) can be put into the form

$$\dot{x}(t) = f_{ti}(x(t), u(t)), \quad (6)$$

with $f_{ti} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. For this specific case, it was shown that ISS implies iISS if f_{ti} is locally Lipschitz and the admissible inputs are locally essentially bounded, i.e. $\mathcal{U} = L^\infty_{loc}(\mathbb{R}^m)$. Also for time-invariant systems, it was shown that the converse implication does not hold. To provide further insight into this relationship, we recall known characterizations of ISS and iISS, and the main argument used to prove that ISS implies iISS. For future reference, we formulate these results employing our notation for families of systems (1).

Definition 2.7. A function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \Lambda \to \mathbb{R}_{\geq 0}$, which is smooth in the first two arguments, and for which there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for all $t_0 \geq 0$, $\xi \in \mathbb{R}^n$ and $\lambda \in \Lambda$,

$$\alpha_1(|\xi|) \leq V(t, \xi, \lambda) \leq \alpha_2(|\xi|), \quad (7)$$

is said to be

(a) a dissipation-form ISS-Lyapunov function for (1) if there exist $\alpha_3 \in \mathcal{K}_\infty$ and $\eta \in \mathcal{K}$ such that the following holds for all $t \geq 0$, $\xi \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$ and $\lambda \in \Lambda$,

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \xi} f_\lambda(t, \xi, \mu) \leq -\alpha_3(|\xi|) + \eta(|\mu|); \quad (8)$$

(b) an implication-form ISS-Lyapunov function for (1) if there exist $\alpha_4, \chi \in \mathcal{K}_\infty$ such that for every $(t, \xi, \mu, \lambda) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \times \Lambda$ for which $|\xi| \geq \chi(|\mu|)$, then

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \xi} f_\lambda(t, \xi, \mu) \leq -\alpha_4(|\xi|); \quad (9)$$

[2] With a finite number of discontinuities in every bounded interval.

[3] Recall the notation $\|u\|_{\gamma_2} = \int_0^\infty \gamma_2(|u(t)|)dt$.

(c) an iISS-Lyapunov function for (1) if there exist $\eta \in \mathcal{K}$ and $\alpha_5 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ continuous and positive definite, such that the following holds for all $t \geq 0$, $\xi \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$ and $\lambda \in \Lambda$,

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \xi} f_\lambda(t, \xi, \mu) \leq -\alpha_5(|\xi|) + \eta(|\mu|). \tag{10}$$

Any of the functions V from Definition 2.7 are said to be time-invariant if they do not depend on t.

By setting $\mathcal{U} = L^\infty_{loc}(\mathbb{R}^m)$ and proceeding as in Sontag and Wang (1995) for ISS and as in Angeli et al. (2000a) for iISS, it can be shown that the existence of a dissipation- or implication-form ISS-Lyapunov function implies that the family of systems is ISS and that the existence of an iISS-Lyapunov function implies that the family is iISS. For time-invariant systems the following results are well-known (see Angeli et al., 2000a and Sontag & Wang, 1995).

Theorem 2.8. *Let $\mathcal{U} = L^\infty_{loc}(\mathbb{R}^m)$. The time-invariant system (6) with f_{ti} locally Lipschitz is*

(i) *ISS if and only if it admits a time-invariant dissipation-form ISS-Lyapunov function.*
(ii) *ISS if and only if it admits a time-invariant implication-form ISS-Lyapunov function.*
(iii) *iISS if and only if it admits a time-invariant iISS-Lyapunov function.*

From inequalities (8) and (10) for the dissipation-form ISS-Lyapunov and iISS-Lyapunov functions, it is easy to see that under the assumptions of Theorem 2.8, ISS implies iISS. Also from Theorem 2.8, it follows that a time-invariant system admits a dissipation-form ISS-Lyapunov function if and only if it admits an implication-form one. Loosely speaking, we may say that existing proofs of the fact that ISS implies iISS require Lyapunov characterizations for both properties. However, extra care must be taken for time-varying systems, as we next show.

2.6. A first obstacle for time-varying systems

For a time-varying system, (1) can be put into the form

$$\dot{x}(t) = f_{tv}(t, x(t), u(t)), \tag{11}$$

with $f_{tv} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. If f_{tv} is locally Lipschitz, it was shown in Edwards et al. (2000) that (11) is ISS if and only if it admits an implication-form ISS-Lyapunov function. Note, however, that for establishing the implication ISS $\Rightarrow$ iISS as explained above, a dissipation-form ISS-Lyapunov function is required. In Edwards et al. (2000), an example of a specific time-varying system that admits an implication-form ISS-Lyapunov function which is not a dissipation-form ISS-Lyapunov function was given. A first contribution of this paper is to show that local Lipschitz continuity of f_{tv} is not sufficient to guarantee that ISS implies iISS. This is established in Proposition 2.9.

Proposition 2.9. *Let $\mathcal{U} = L^\infty_{loc}(\mathbb{R}^m)$. There exists a system of the form (11), with $f_{tv} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ locally Lipschitz, such that it is ISS but not iISS.*

Proof. Consider the one-dimensional single-input time-varying system considered in Edwards et al. (2000),

$$\dot{x} = -x + (1+t)g(u - |x|) =: f_{tv}(t, x, u), \tag{12}$$

with $f_{tv} : \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ defined as follows: $g(s) = 0$ for all $s \leq 0$ and $g(s) = s$ for all $s > 0$. It is clear that f_{tv} is locally Lipschitz in (t, x, u). The function $V(t, \xi, \lambda) = \bar{V}(\xi) = \xi^2$ is a time-invariant implication-form ISS-Lyapunov function for system (12), because it satisfies (7) with $\alpha_1(r) = \alpha_2(r) = r^2$, and

$$\bar{V}(\xi) \geq |\mu|^2 \Longrightarrow \nabla\bar{V}(\xi) f_{tv}(t, \xi, \mu) = -2\bar{V}(\xi).$$

Then system (12) is ISS.

Next, we show that the system is not iISS. Suppose for a contradiction that it is iISS, and let $\beta \in \mathcal{KL}$, and $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ characterize the iISS property, as in (5). Let $M := 4\gamma_1(1)$. For $t_0 \geq 1$ consider $u_{t_0} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined as follows: $u_{t_0}(t) = 0$ if $t < t_0$ and $u_{t_0}(t) = M$ if $t \geq t_0$. Let $x_{t_0}(\cdot) = x(\cdot, t_0, 0, u_{t_0})$. Since $\dot{x}_{t_0}$ is continuous and $\dot{x}_{t_0}(t_0) = f_{tv}(t_0, 0, M) = (1+t_0)M > 0$, then $\dot{x}_{t_0}$ is positive on some interval (t_0, t') with $t' > t_0$. Let $T = \sup\{t : t > t_0,\ 0 < \dot{x}_{t_0}(s)\ \forall s \in (t_0, t)\}$. We have that $T > t_0$, $\dot{x}_{t_0}(s) > 0$ for all $s \in (t_0, T)$ and that $\dot{x}_{t_0}(T) = 0$ if T is finite. Since $f_{tv}(t, \xi, M) \geq M/2$ for all $0 \leq \xi \leq M/2$ and $t \geq 1$, and x is strictly increasing in $[t_0, T)$, there exists a unique $\tau_{t_0} \in (t_0, T)$ such that $x_{t_0}(\tau_{t_0}) = M/2$. Since $0 \leq x_{t_0}(t) \leq M/2$ for all $t \in [t_0, \tau_{t_0}]$, we have that

$$\dot{x}_{t_0}(t) \geq \frac{Mt}{2} \quad \forall t \in [t_0, \tau_{t_0}].$$

Integrating the above inequality from t_0 to τ_{t_0}, we reach

$$\frac{M}{2} = x_{t_0}(\tau_{t_0}) \geq \frac{M}{4}(\tau_{t_0}^2 - t_0^2) = \frac{M}{4}(\tau_{t_0} - t_0)(\tau_{t_0} + t_0).$$

Consequently, $(\tau_{t_0} - t_0)(\tau_{t_0} + t_0) \leq 2$, and therefore $\tau_{t_0} - t_0 \leq 2/(\tau_{t_0} + t_0) \leq 2/t_0$. If we take $t_0 := 2\gamma_2(M)$ with $x_{t_0}(t_0) = 0$, then we have that $\tau_{t_0} - t_0 \leq 1/\gamma_2(M)$ and that

$$2\gamma_1(1) = |x_{t_0}(\tau_{t_0})|$$
$$\leq \beta(|x(t_0)|, \tau_{t_0} - t_0) + \gamma_1\left(\int_{t_0}^{\tau_{t_0}} \gamma_2(M)\, ds\right) \leq \gamma_1(1).$$

We have thus arrived to a contradiction. This establishes that the system is not iISS. ∎

Remark 2.10. The system (12) can be posed as a time-invariant system by treating t as a state variable and considering the extended system formed by (12) and $\dot{t} = 1$. Then, the ISS and iISS properties of (12) coincide with the ISS and iISS of the extended system with respect to the two measures $\omega_1(x, t) = \omega_2(x, t) = |x|$, as per Definitions 1 and 4 of Kellett et al. (2013). In this context, the system in the proof of Proposition 2.9 is an example of the fact that ISS with respect to two measures does not imply iISS with respect to two measures. This example seems to be simpler than Example 2 in Section 4.2 of Kellett et al. (2013). ∘

In the next section, we will show that appropriate assumptions on the functions f_λ indeed ensure that ISS $\Rightarrow$ iISS for parametrized time-varying control systems.

3. Main results: ISS implies iISS

In this section, we prove that under suitable assumptions, it is true that ISS implies iISS in the general context considered. Three interesting features of the proof we give are: (a) it does not rely on Lyapunov characterizations; (b) the set of admissible inputs $\mathcal{U}$ for which the family of systems is proved to be iISS is larger than $L^\infty_{loc}(\mathbb{R}^m)$; (c) it provides a relationship between the ISS and iISS gains in terms of bounds on the system functions f_λ. Existing proofs of the fact that ISS implies iISS can also provide relationships between the aforementioned gains, but only in terms of the comparison functions appearing in the Lyapunov characterizations.

In Section 3.1, we state the assumptions required, our main result, and provide comments on the assumptions. Worthy of mention is the fact that Lipschitz continuity with respect to the input variable is not required as an assumption. The proof of our main result is given along Sections 3.2–3.4. In Section 3.2, we derive an equivalent formulation for the required assumptions. In Section 3.3, we show that an existing characterization of iISS, valid for switched systems, also holds for families of systems. In Section 3.4, we provide the main proof, employing the results in the previous subsections. Discussion and further comments are given in Section 3.5.

3.1. Assumptions and main theorem

The assumptions required are given next.

Assumption 1. The functions f_λ in (1) satisfy:

(A1) for every $\lambda \in \Lambda$, $f_\lambda(\cdot, \xi, \mu)$ is Lebesgue measurable for all $(\xi, \mu) \in \mathbb{R}^n \times \mathbb{R}^m$.

(A2) For every $r, s \geq 0$, there exists $L_1 = L_1(r, s) \geq 0$ such that

$$|f_\lambda(t, \xi_1, \mu) - f_\lambda(t, \xi_2, \mu)| \leq L_1|\xi_1 - \xi_2| \tag{13}$$

for all $t \geq 0$, $\xi_1, \xi_2 \in B_r^n$, $\mu \in B_s^m$ and $\lambda \in \Lambda$.

(A3) There exists $\omega \in \mathcal{K}_\infty$ and for every $r, s \geq 0$, there exists $L_2 = L_2(r, s) \geq 0$ such that

$$|f_\lambda(t, \xi, \mu_1) - f_\lambda(t, \xi, \mu_2)| \leq L_2\omega(|\mu_1 - \mu_2|) \tag{14}$$

for all $t \geq 0$, $\xi \in B_r^n$, $\mu_1, \mu_2 \in B_s^m$ and $\lambda \in \Lambda$.

Remark 3.1. Without loss of generality, L_1 and L_2 in Assumption 1 can be assumed continuous and nondecreasing in each variable. ∘

Remark 3.2. Consider the switched system (2) with $f_{sw} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \times \Gamma \to \mathbb{R}^n$ such that the family of functions f_{sw_i}, $i \in \Gamma$, where $f_{sw_i}(\cdot, \cdot, \cdot) := f_{sw}(\cdot, \cdot, \cdot, i)$, satisfies the conditions in Assumption 1, with i in place of λ and Γ instead of Λ (i.e. as would be posed within a switched system context). Then, we show that for any subset $\mathcal{S}$ of admissible switching signals, the parametrized family of time-varying systems

$$\dot{x}(t) = f_\sigma(t, x(t), u(t)), \quad \sigma \in \mathcal{S},$$

with f_σ defined as in (3), satisfies Assumption 1. From the definition of f_σ and the fact that f_{sw_i} satisfies (A2) and (A3) for all $i \in \Gamma$, it easily follows that f_σ satisfies (A2) and (A3) for all $\sigma \in \mathcal{S}$. In order to show that f_σ satisfies (A1), define $\tau_0 := 0$ and consider the strictly increasing sequence $\{\tau_k\}_{k=1}^N$ of switching times of σ, where N may be finite or ∞. Define the sequence $\{i_k\}_{k=0}^N$ through $i_k := \sigma(\tau_k)$. If $N = \infty$, then $\tau_k \to \infty$ (because σ has a finite number of discontinuities in every bounded interval). For each $k \in \mathbb{N}_0$, let $g_k(t) = 1$ if $\tau_k \leq t < \tau_{k+1}$ and $g_k(t) = 0$ otherwise. Then, for each (ξ, μ), we have $f_\sigma(t, \xi, \mu) = \sum_{k=0}^{\infty} g_k(t) f_{sw}(t, \xi, \mu, i_k)$ for all $t \geq 0$. Since $g_k(\cdot) f_{sw}(\cdot, \xi, \mu, i_k)$ is Lebesgue measurable for each k, the measurability of $f_\sigma(\cdot, \xi, \mu)$ follows. The case when N is finite can be proved in a similar way. ∘

Our main result is the following.

Theorem 3.3. *Consider a family (1) for which Assumption 1 holds. Suppose that the family is ISS. Then there exists $\hat{\chi} \in \mathcal{K}_\infty$ such that $\mathcal{U} = L^1_{loc,\hat{\chi}}(\mathbb{R}^m)$ is a set of admissible inputs and the family is iISS with iISS gain $\hat{\chi}$.*

The proof of Theorem 3.3 is given along the next subsections. The construction of suitable iISS gains will be explained later, in Section 3.4. We next provide some comments on the required Assumption 1. First, note that (A2) imposes local Lipschitz continuity of f_λ in (1) with respect to the state variable, uniformly over every $t \geq 0$ and $\lambda \in \Lambda$, and also over state and input values in compact sets. By contrast, (A3) does not require Lipschitz continuity with respect to the input variable although a type of uniform continuity is indeed required. In the case of a single time-invariant system, and since local Lipschitz continuity with respect to the input variable is not required, then Assumption 1 is weaker than requiring local Lipschitz continuity of f_{ti} in (6). The assumptions of Theorem 3.3 are thus weaker than existing ones for time-invariant systems.

3.2. Equivalent assumptions

We next show that Assumption 1 can be equivalently formulated in an apparently very different manner. Establishing this equivalent formulation is important since it will both simplify the proof of Theorem 3.3 and allow the construction of iISS gains. The proof of Lemma 3.4 is given in Appendix A.

Lemma 3.4. *The functions f_λ in (1) satisfy (A2)–(A3) of Assumption 1 if and only if they satisfy (B1)–(B2):*

(B1) There exists $\tilde{\gamma} \in \mathcal{K}_\infty$ and nondecreasing and continuous functions $N, O : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that[4]

$$|f_\lambda(t, \xi, \mu_1) - f_\lambda(t, \xi, \mu_2)| \leq \tilde{\gamma}(|\mu_1 - \mu_2|)\big[N(|\xi|) + O(|\mu_1| \wedge |\mu_2|)\big]$$

holds for all $t \geq 0$, $\xi \in \mathbb{R}^n$, $\mu_1, \mu_2 \in \mathbb{R}^m$ and $\lambda \in \Lambda$.

(B2) There exists $\eta, \gamma \in \mathcal{K}_\infty$, and $P : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ nondecreasing and continuous, such that

$$\limsup_{s \to 0^+} \frac{\eta(s)}{s} < \infty \tag{15}$$

and for all $t \geq 0$, $\xi_1, \xi_2 \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$, and $\lambda \in \Lambda$,

$$|f_\lambda(t, \xi_1, \mu) - f_\lambda(t, \xi_2, \mu)| \leq \eta(|\xi_1 - \xi_2|)[P(|\xi_1| \wedge |\xi_2|) + \gamma(|\mu|)].$$

Remark 3.5. Since $\eta \in \mathcal{K}_\infty$ in (B2), from (15) it follows that for every $M > 0$ there exists $L = L(M)$ so that

$$\eta(s) \leq Ls \quad \text{for all } 0 \leq s \leq M, \tag{16}$$

where the function $L(\cdot)$ can be taken continuous and nondecreasing. ∘

The following lemma, whose proof is provided in Appendix B, gives a bound on f_λ that will be required later and a first result on the sets of admissible inputs.

Lemma 3.6. *Let Assumption 1 hold, and consider $\gamma, \tilde{\gamma} \in \mathcal{K}_\infty$ from Lemma 3.4. Then,*

(i) There exists a nondecreasing function $\hat{N} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ such that $|f_\lambda(t, \xi, \mu)| \leq \hat{N}(|\xi|)(1 + \tilde{\gamma}(|\mu|))$ for all $t \geq 0$, all $\xi \in \mathbb{R}^n$, all $\mu \in \mathbb{R}^m$ and all $\lambda \in \Lambda$.

[4] Recall the notation $a \wedge b = \min\{a, b\}$.

(ii) *For every $\chi \in \mathcal{K}_\infty$ such that $\chi \geq \max\{\gamma, \tilde{\gamma}\}$, $L^1_{loc,\chi}(\mathbb{R}^m)$ is a set of admissible inputs for (1).*

3.3. An existing characterization of iISS

The proof of our main result is based on a characterization of the iISS property which was obtained in Haimovich and Mancilla-Aguilar (2018). This characterization essentially says that iISS is equivalent to zero-input global uniform asymptotic stability in combination with uniform bounded-energy input/bounded state (Angeli et al., 2000b), even for (families of) time-varying systems. We next recall the corresponding definitions.

Definition 3.7. Let $\mathcal{U}$ be a set of admissible inputs for the family of systems (1). The family is said to be uniformly bounded-energy input/bounded state (UBEBS), if there exist functions $\alpha, \phi, \gamma_2 \in \mathcal{K}_\infty$ (γ_2 will be referred to as an UBEBS gain), and a constant $c \geq 0$, such that for every $t_0 \geq 0$, $\xi \in \mathbb{R}^n$, $u \in \mathcal{U}$ and $\lambda \in \Lambda$, $x(\cdot) = x(\cdot, t_0, \xi, u, \lambda)$ satisfies the following estimate for all $t \in I_{t_0,\xi,u,\lambda}$,

$$|x(t)| \leq \alpha(|x(t_0)|) + \phi\left(\|u\|_{\gamma_2}\right) + c. \tag{17}$$

Remark 3.8. Analogously to Remark 2.5, an equivalent definition of UBEBS is obtained if u is replaced by $u_{[t_0,t]}$ in (17), and the solutions of (1) are forward complete for all inputs $u \in L^1_{loc,\gamma_2}(\mathbb{R}^m) \cap \mathcal{U}$. ∘

Definition 3.9. The family (1) is said to be zero-input globally uniformly asymptotically stable (0-GUAS) if there exists $\beta \in \mathcal{KL}$ such that for every $t_0 \geq 0$, $\xi \in \mathbb{R}^n$ and $\lambda \in \Lambda$, $x(\cdot) = x(\cdot, t_0, \xi, \mathbf{0}, \lambda)$, where $\mathbf{0}$ denotes the input $u \in \mathcal{U}$ such that $u(s) = 0$ for all $s \geq 0$, verifies

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0) \quad \text{for all } t \geq t_0 \geq 0. \tag{18}$$

Theorem 3.10. *Let the functions f_λ in (1) satisfy Assumption 1, and consider $\gamma, \tilde{\gamma} \in \mathcal{K}_\infty$ from Lemma 3.4. Let $\chi \in \mathcal{K}_\infty$ satisfy $\chi \geq \max\{\gamma, \tilde{\gamma}\}$, and set $\mathcal{U} = L^1_{loc,\chi}(\mathbb{R}^m)$. Then, if the family (1) is 0-GUAS and UBEBS with UBEBS gain γ_2, then it is iISS with iISS gain $\max\{\chi, \gamma_2\}$.*

Proof. First, note that by Lemma 3.6(ii), $\mathcal{U}$ is a set of admissible inputs. In Section 2.3 we have shown that a switched system can be posed as a parametrized family of systems. This proof is based on the converse: posing a parametrized family as a switched system, and applying the results in Haimovich and Mancilla-Aguilar (2018). This converse formulation is possible because the index set over which the switching signal takes values in Haimovich and Mancilla-Aguilar (2018) is just any arbitrary nonempty set, and hence it can be, e.g., infinite and uncountable. Consider the (possibly infinite, uncountable) index set $\Gamma := \Lambda$, and set $f_{sw}(t, \xi, \mu, \lambda) := f_\lambda(t, \xi, \mu)$. Define the set of admissible switching signals, $\mathcal{S}$, as the set of all the constant functions $\sigma : \mathbb{R}_{\geq 0} \to \Lambda$. Then, the resulting family of time-varying systems (1) is iISS, UBEBS or 0-GUAS (according to Definitions 2.4, 3.7 or 3.9) if and only if (2), with inputs in $\mathcal{U}$ is, respectively, iISS w.r.t. $\mathcal{S}$, UBEBS w.r.t. $\mathcal{S}$ or 0-GUAS w.r.t. $\mathcal{S}$, (according to Definitions 1, 4 or 3 in Haimovich and Mancilla-Aguilar (2018)).

Note that $f_\lambda(t, 0, 0) = 0$ for every $\lambda \in \Gamma$ and hence the blanket assumption in Haimovich and Mancilla-Aguilar (2018) is satisfied. Assumption 1 of Haimovich and Mancilla-Aguilar (2018) also is satisfied. In fact, condition (C1) in Assumption 1 of Haimovich and Mancilla-Aguilar (2018) coincides with (i) in Lemma 3.6, while (C2) and (C3) in such an assumption straightforwardly follow from (B1) and (B2), respectively, the latter taking Remark 3.5 into account. Therefore, the switched system (2) satisfies the assumptions of Theorem 1 in Haimovich and Mancilla-Aguilar (2018). We must remark that Haimovich and Mancilla-Aguilar (2018) assumes throughout that the set of admissible inputs is $L^\infty_{loc}(\mathbb{R}^m)$. Nevertheless, this assumption was only meant to guarantee existence (but not uniqueness) of solutions of the corresponding switched system. Under the current assumptions, solutions not only exist but also are unique, and hence Theorem 1 of Haimovich and Mancilla-Aguilar (2018) remains valid in the current context. The proof then follows from item b) of Theorem 1 of Haimovich and Mancilla-Aguilar (2018), taking into account that γ from Theorem 1 of Haimovich and Mancilla-Aguilar (2018) corresponds to $\tilde{\gamma}$ of Lemma 3.4. ■

3.4. A direct proof that ISS implies iISS

We shall prove that ISS implies iISS by showing that ISS implies UBEBS, and applying Theorem 3.10. We require the following technical result, whose proof is given in Appendix C.

Lemma 3.11. *Let $g_1 : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ and $g_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be continuous. Then, there exist $\kappa, \phi \in \mathcal{K}_\infty$ and $c \geq 0$ such that*

$$\kappa(r) \geq g_1(r) g_2(s), \quad \textit{for all } r \geq \phi(s) + c \textit{ and } s \geq 0. \tag{19}$$

For proving that ISS implies UBEBS and to keep track of the UBEBS gain, we need to define some auxiliary functions.

Suppose that the family (1) is ISS and that the functions f_λ satisfy Assumption 1. Let $\beta \in \mathcal{KL}$ and $\rho \in \mathcal{K}_\infty$ characterize the ISS property and let $N, O, P, \eta, \gamma, \tilde{\gamma}$ be given by Lemma 3.4. Define $h_1, h_2 : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ via

$$h_1(r, b) := N(\beta(r, 0) + \rho(b)) + O(b), \tag{20}$$

$$h_2(r, b) := P(\beta(r, 0) + \rho(b)). \tag{21}$$

Let $L : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be continuous, nondecreasing, and such that for every $M \geq 0$, (16) holds with $L = L(M)$. In correspondence with every $r > 0$, define $T_r \geq 0$ continuous and such that

$$\beta(r, T_r) \leq r/3. \tag{22}$$

Define also

$$b_r := \rho^{-1}(r/3), M_r := r/3, L_r := L(M_r). \tag{23}$$

Define $g_1 : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ and $g_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ via

$$g_1(r) = \frac{h_1(r, b_r) e^{L_r T_r h_2(r, b_r) + L_r^2/2}}{M_r}, \tag{24}$$

$$g_2(s) = s e^{s^2/2}. \tag{25}$$

Let κ be given by Lemma 3.11 for g_1, g_2 as above. Finally, define $\alpha \in \mathcal{K}_\infty$ via

$$\alpha(b) = \kappa(3\rho(b)). \tag{26}$$

The following theorem establishes that under Assumption 1, ISS implies UBEBS.

Theorem 3.12. *Consider the family (1) and let Assumption 1 hold. Suppose that the family is ISS. Let $\gamma, \tilde{\gamma}$ be given by Lemma 3.4 and let α be the function defined in (26). Let $\chi, \tilde{\chi} \in \mathcal{K}_\infty$ satisfy $\tilde{\chi} \geq \max\{\gamma, \tilde{\gamma}\}$ and $\chi \geq \max\{\gamma, \tilde{\gamma}^2, \alpha^2\}$. Set $\mathcal{U} = L^1_{loc,\tilde{\chi}}(\mathbb{R}^m)$ as the set of admissible inputs. Then, the family of systems is UBEBS with UBEBS gain χ.*

Proof. First, note that according to Lemma 3.6(ii), then $\mathcal{U} = L^1_{loc,\tilde{\chi}}(\mathbb{R}^m)$ is a set of admissible inputs. Given an input $u \in \mathcal{U}$ and a constant $b \geq 0$, let u_b denote a new input, defined as follows:

$$u_b(t) = \begin{cases} \dfrac{b u(t)}{|u(t)|} & \text{if } t \in \Omega_u(b), \\ u(t) & \text{otherwise,} \end{cases} \tag{27}$$

$$\Omega_u(b) := \{t \geq 0 : |u(t)| > b\}. \tag{28}$$

Note that $|u_b(t)| = \min\{|u(t)|, b\}$ for all $t \geq 0$ and hence $\|u_b\| \leq b$

Let $t_0 \geq 0$, $\xi \in \mathbb{R}^n$ and $\lambda \in \Lambda$, and consider an input $u \in \mathcal{U}$ such that

$$\|u\|_\chi = \int_0^\infty \chi(|u(s)|)ds =: E < \infty. \tag{29}$$

Let $x(\cdot) = x(t, t_0, \xi, u, \lambda)$ and define $\tilde{\alpha} \in \mathcal{K}_\infty$ via

$$\tilde{\alpha}(r) = \beta(r, 0) + \frac{2r}{3}, \tag{30}$$

where β is the class-$\mathcal{KL}$ functions which characterizes the ISS property. Let $T_r \geq 0$, $r \geq 0$, be continuous and satisfying (22). Let $\kappa, \phi \in \mathcal{K}_\infty$ and c be given by Lemma 3.11 for the functions g_1 and g_2 defined in (24) and (25) respectively. We can assume, without loss of generality, that $c > 0$.

Claim 1. *Let r be any real number such that $r \geq c + \phi(E)$ and $|x(t_0)| \leq r$, then*

$$|x(t)| \leq \tilde{\alpha}(r) \quad \forall t \geq t_0 \tag{31}$$

Proof of Claim 1. For a fixed $b \geq 0$, let $x_b(\cdot) := x(\cdot, t_0, x(t_0), u_b, \lambda)$, and $\Delta x = x - x_b$. Since (1) is ISS, then

$$|x_b(t)| \leq \beta(|x(t_0)|, t - t_0) + \rho(\|u_b\|) \leq \beta(r, 0) + \rho(b)$$

for all $t \geq t_0$. From (1) and Lemma 3.4, it follows that

$$|\Delta x(t)| \leq \int_{t_0}^t \Big|f_\lambda(s, x(s), u(s)) - f_\lambda(s, x_b(s), u_b(s))\Big|ds$$
$$\leq \int_{t_0}^t \Big|f_\lambda(s, x(s), u(s)) - f_\lambda(s, x_b(s), u(s))\Big|ds$$
$$+ \int_{t_0}^t \Big|f_\lambda(s, x_b(s), u(s)) - f_\lambda(s, x_b(s), u_b(s))\Big|ds$$
$$\leq \int_{t_0}^t \eta(|\Delta x(s)|)[P(|x(s)| \wedge |x_b(s)|) + \gamma(|u(s)|)]ds$$
$$+ \int_{t_0}^t \tilde{\gamma}(|u(s) - u_b(s)|)[N(|x_b(s)|) + O(|u(s)| \wedge |u_b(s)|)]ds$$

holds for all $t \geq t_0$ for which $x(t)$ exists. Then, for all $t \geq t_0$ for which $x(t)$ exists,

$$|\Delta x(t)| \leq \int_{t_0}^t \eta(|\Delta x(s)|)[h_2(r, b) + \gamma(|u(s)|)]ds$$
$$+ h_1(r, b) \int_{t_0}^t \tilde{\gamma}(|u(s) - u_b(s)|)ds \tag{32}$$

For $t \geq t_0$, we have the following inequalities:

$$\int_{t_0}^t \tilde{\gamma}(|u(s) - u_b(s)|)ds \leq \int_{\Omega_u(b)} \tilde{\gamma}(|u(s)| - b)ds$$
$$\leq \int_{\Omega_u(b)} \tilde{\gamma}(|u(s)|)ds$$

Applying the Schwarz inequality, then

$$\int_{\Omega_u(b)} \tilde{\gamma}(|u(s)|)ds \leq |\Omega_u(b)|^{1/2} \sqrt{\int_{\Omega_u(b)} \tilde{\gamma}^2(|u(s)|)ds}$$
$$\leq |\Omega_u(b)|^{1/2}\sqrt{E},$$

where we have used the fact that $\chi \geq \tilde{\gamma}^2$ and where $|\Omega_u(b)|$ denotes the Lebesgue measure of the set $\Omega_u(b)$. Also, we have

$$E \geq \int_{\Omega_u(b)} \chi(|u(s)|)ds \geq |\Omega_u(b)|\chi(b),$$

and hence

$$|\Omega_u(b)| \leq \frac{E}{\chi(b)}, \quad \text{if } b > 0.$$

Combining the obtained inequalities, we reach, for $b > 0$,

$$\int_{t_0}^t \tilde{\gamma}(|u(s) - u_b(s)|)ds \leq \frac{E}{\sqrt{\chi(b)}} \leq \frac{E}{\alpha(b)}, \tag{33}$$

where we have used the fact that $\chi \geq \alpha^2$. Let b_r, M_r be in correspondence with r as defined in (22)–(23) and consider $b = b_r$. Define

$$\tau := \inf\{t \geq t_0 : |\Delta x(t)| \geq M_r\}.$$

We next show that $\tau \geq t_0 + T_r$. From the definition of τ and the continuity of Δx, we have

$$|\Delta x(t)| \leq M_r, \quad \text{for all } t_0 \leq t \leq \tau.$$

From (16), then $\eta(|\Delta x(t)|) \leq L_r|\Delta x(t)|$ for all $t_0 \leq t \leq \tau$. From (32) and (33), then

$$|\Delta x(t)| \leq \frac{h_1(r, b_r)E}{\alpha(b_r)} + \int_{t_0}^t [h_2(r, b_r) + \gamma(|u(s)|)]L_r|\Delta x(s)|ds$$

for all $t_0 \leq t \leq \tau$. Applying Gronwall's inequality, we reach

$$|\Delta x(t)| \leq \frac{h_1(r, b_r)E}{\alpha(b_r)} e^{\int_{t_0}^t [h_2(r,b_r)+\gamma(|u(s)|)]L_r ds}$$
$$\leq \frac{h_1(r, b_r)E}{\alpha(b_r)} e^{L_r[(t-t_0)h_2(r,b_r)+E]},$$

where we have used the fact that $\chi \geq \gamma$. Using the inequality $\theta\varphi \leq (\theta^2 + \varphi^2)/2$, it follows that

$$\frac{h_1(r, b_r)E}{\alpha(b_r)} e^{L_r[T_r h_2(r,b_r)+E]} \leq \frac{g_1(r)g_2(E)}{\alpha(b_r)} M_r$$
$$= \frac{g_1(r)g_2(E)}{\kappa(r)} M_r \leq M_r$$

because $r \geq c + \phi(E)$. Therefore, $\tau \geq t_0 + T_r$ and hence $|\Delta x(t)| \leq M_r$ for all $t_0 \leq t \leq t_0 + T_r$. The solution x can be bounded as follows

$$|x(t)| \leq |x_{b_r}(t)| + |\Delta x(t)| \leq \beta(r, t - t_0) + \rho(b_r) + M_r$$
$$\leq \beta(r, 0) + \rho(b_r) + M_r = \tilde{\alpha}(r),$$

for all $t_0 \leq t \leq t_0 + T_r$. In particular, at $t_1 := t_0 + T_r$, we also have

$$|x(t_1)| \leq \beta(r, T_r) + \rho(b_r) + M_r \leq r.$$

By defining the sequence $t_i = t_0 + iT_r$, $i \in \mathbb{N}_0$ and applying recursively the precedent reasoning to t_i in place of t_0, we obtain

$$|x(t)| \leq \tilde{\alpha}(r) \quad \forall t \in [t_i, t_{i+1}]$$
$$|x(t_{i+1})| \leq r.$$

This concludes the proof of the claim. ◦

Next, we will prove that

$$|x(t)| \leq \max\{\tilde{\alpha}(|x(t_0)|), \tilde{\alpha}(\phi(E) + c)\}.$$

If $|x(t_0)| \geq \phi(E)+c$, by applying Claim 1 with $r = |x(t_0)|$ it follows that $|x(t)| \leq \tilde{\alpha}(|x(t_0)|)$ for all $t \geq t_0$.

If $|x(t_0)| < \phi(E) + c$, let $t_1 = \inf\{t \geq t_0 : |x(t)| \geq \phi(E) + c\}$. If $t_1 = \infty$, then $|x(t)| < \phi(E) + c \leq \tilde{\alpha}(\phi(E) + c)$ for all $t \geq t_0$. If t_1 is finite, then $|x(t)| < \phi(E) + c \leq \tilde{\alpha}(\phi(E) + c)$ for all $t \in [t_0, t_1)$ and $|x(t_1)| = \phi(E) + c$. By applying Claim 1 with t_1 instead of t_0 and $r := \phi(E) + c$ we obtain $|x(t)| \leq \tilde{\alpha}(\phi(E) + c)$ for all $t \geq t_1$. Therefore $|x(t)| \leq \tilde{\alpha}(\phi(E) + c)$ for all $t \geq t_0$.

Since for all $t \geq t_0$

$$
\begin{aligned}
|x(t)| &\leq \max\{\tilde{\alpha}(|x(t_0)|), \tilde{\alpha}(\phi(E) + c)\} \\
&\leq \tilde{\alpha}(|x(t_0)|) + \tilde{\alpha}(\phi(E) + c) \\
&\leq \tilde{\alpha}(|x(t_0)|) + \tilde{\alpha}(2\phi(E)) + \tilde{\alpha}(2c)
\end{aligned}
$$

it follows that (1) is UBEBS with UBEBS gain χ. ■

We can now give the proof of our main result.

Proof of Theorem 3.3. Since the family (1) is ISS, then it is 0-GUAS. Let $\gamma, \tilde{\gamma}, \alpha$ be as in the statement of Theorem 3.12, and let $\hat{\chi} \in \mathcal{K}_\infty$ satisfy $\hat{\chi} \geq \max\{\gamma, \tilde{\gamma}, \tilde{\gamma}^2, \alpha^2\}$. By Theorem 3.12, it follows that setting $\mathcal{U} = L^1_{\mathrm{loc},\hat{\chi}}(\mathbb{R}^m)$ as the set of admissible inputs, (1) is UBEBS with UBEBS gain $\hat{\chi}$. By Theorem 3.10 it then follows that (1) is iISS with iISS gain $\hat{\chi}$. This concludes the proof of Theorem 3.3. ■

3.5. Discussion

We next provide some brief comments on our results and proofs. Theorem 3.3 shows not only that under Assumption 1 ISS implies iISS but also that inputs need not be constrained to be locally essentially bounded functions in order for the iISS property to hold, provided the system equation solutions exist and are unique.

The main idea in the proof of Theorem 3.3 is to show that ISS implies UBEBS. This is established in Theorem 3.12. The proof of Theorem 3.12 is based on establishing a bound on the difference between the trajectories generated by an arbitrary input in the set of admissible inputs and a bounded input constructed in correspondence with the former. Lipschitz continuity of f_λ with respect to the state variable is required in order to apply Gronwall's inequality. An interesting question is whether the implication ISS $\Rightarrow$ iISS also holds when solutions are not necessarily unique. If this were the case, a suitable proof should avoid the need to employ Gronwall's inequality.

The relationships between the ISS, iISS, UBEBS and 0-GUAS properties can be summarized as follows: ISS $\Longrightarrow$ 0-GUAS + UBEBS $\Longleftrightarrow$ iISS, where the first implication follows under Assumption 1 and the if and only if is established in Haimovich and Mancilla-Aguilar (2018).

4. Example

4.1. SQZ-source inverter model

Consider the ideal switched model of the semi-quasi-Z-source inverter (Cao, Jiang, Yu, & Peng, 2011; Haimovich, Middleton, & De Nicoló, 2013), connected to a cubic-law time-varying resistive load and under time-varying input voltage u, similarly to Example 4.1 in Mancilla-Aguilar, Haimovich, and García (2017):

$$\dot{x}(t) = f_{\mathrm{sw}}(t, x(t), u(t), \sigma(t))$$

with $f_{\mathrm{sw}} : \mathbb{R}_{\geq 0} \times \mathbb{R}^4 \times \mathbb{R} \times \{1, 2\} \to \mathbb{R}^4$,

$$
\begin{aligned}
& f_{\mathrm{sw}}(t, \xi, \mu, i) = A_i \xi - e_4 g_i(t, e_4' \xi) + b_i \mu, \\
& e_4 = [0\ 0\ 0\ 1]', \qquad P = \mathrm{diag}(\bar{L}_1, \bar{L}_2, \bar{C}_1, \bar{C}_2), \\
& A_1 = P^{-1} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad A_2 = P^{-1} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \\
& b_1 = \left[1/\bar{L}_1\ 0\ 0\ 0\right], \qquad b_2 = \left[0\ -1/\bar{L}_2\ 0\ 0\right], \\
& g_i(t, v) = \frac{G_i(t)}{\bar{C}_2} v^3, \quad G_i(t) = |\cos(t^2 + a_i)| + \epsilon_i,
\end{aligned}
$$

for some $a_i \in \mathbb{R}$ and $\epsilon_i > 0$, for $i = 1, 2$. The positive constants $\bar{L}_1, \bar{L}_2, \bar{C}_1, \bar{C}_2$ represent the inverter inductance and capacitance values. Irrespective of the load function g_i, stability of this inverter model requires constantly switching between modes $\sigma(t) = 1$ and $\sigma(t) = 2$, and imposing additional restrictions on the time spent in mode 2 (De Nicoló, Haimovich, & Middleton, 2016). Let $\mathcal{S}$ denote the set of switching signals $\sigma : \mathbb{R}_{\geq 0} \to \{1, 2\}$ where each mode has minimum ($d_{\min}$) and maximum ($d_{\max}$) dwell-times satisfying $0 < d_{\min} < d_{\max} < \pi\sqrt{\bar{L}_1 \bar{C}_1}$. Following the steps in Example 4.1 of Mancilla-Aguilar et al. (2017), this switched system can be shown to be ISS (uniformly) w.r.t. $\mathcal{S}$ (as per Definition 4.1 in Mancilla-Aguilar et al. (2017)) by means of the weak quadratic Lyapunov-type function $V(x) = \frac{1}{2} x' P x$.

We next pose this switched system as a family of parametrized systems, as explained in Section 2.3. Hence, we take $\mathcal{S}$ as the set of parameters and for every $\sigma \in \mathcal{S}$ define $f_\sigma : \mathbb{R}_{\geq 0} \times \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$ via (3). The family

$$\dot{x}(t) = f_\sigma(t, x(t), u(t)), \quad \sigma \in \mathcal{S} \tag{34}$$

thus defined is an equivalent formulation of the original switched system and is then ISS, according to Definition 2.4.

4.2. SQZ-source inverter under input perturbations

Let the input voltage u be decomposed as $u = u_0 + \Delta u$, where u_0 is the (possibly time-varying) nominal input voltage and $\Delta u(t)$ is a perturbation, expressed as the output of a time-varying system with inputs v:

$$\dot{z}(t) = g(t, z(t), v(t)), \tag{35}$$

$$\Delta u(t) = h(t, z(t)), \tag{36}$$

with $g : \mathbb{R}_{\geq 0} \times \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k$ and $h : \mathbb{R}_{\geq 0} \times \mathbb{R}^k \to \mathbb{R}$. Let the function g satisfy $g(t, 0, 0) = 0$ for all $t \geq 0$ and Assumption 1, and let h be measurable in t, locally Lipschitz in z uniformly in $t \geq 0$, and satisfy $h(t, 0) = 0$ for all $t \geq 0$. Consider the cascade connection of (35)–(36) with (34) through $u = u_0 + \Delta u$. The resulting connection can be expressed as a family of systems with state $\mathbf{x} := \mathrm{col}[x, z]$, input $\mathbf{w} := \mathrm{col}[u_0, v]$, and equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}_\sigma(t, \mathbf{x}(t), \mathbf{w}(t)), \quad \sigma \in \mathcal{S}, \tag{37}$$

with $\mathbf{f}_\sigma = \mathrm{col}[f^1_\sigma, f^2_\sigma]$ and

$$f^1_\sigma(t, \mathbf{x}, \mathbf{w}) := f_\sigma(t, x, u_0 + h(t, z)), \tag{38}$$

$$f^2_\sigma(t, \mathbf{x}, \mathbf{w}) := g(t, z, v). \tag{39}$$

We would like to know whether (37) is iISS, since this would mean that the state $\mathbf{x}$ cannot diverge even under specific unbounded nominal and perturbation inputs, whose energy is finite when measured according to an iISS gain.

We stress that finding an iISS-Lyapunov function for (37) is an extremely hard task. To see why this is so, suppose that $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n+k} \times \Lambda \to \mathbb{R}_{\geq 0}$ is an iISS-Lyapunov function for (37). Then, $V_1(t, x, \sigma) := V(t, \mathrm{col}[x, 0], \sigma)$ must be an iISS-Lyapunov function for the family (1) with f_λ defined as explained. The function $V_1(t, x, \sigma)$ cannot depend only on the values of the switching signals σ, i.e. V_1 cannot be of the simple form $V_1(t, x, \sigma) = W(t, x, \sigma(t))$ for some function $W : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \{1, 2\} \to \mathbb{R}$. In fact, since V_1 has to be differentiable at any $t \geq 0$ and the switching signal σ may have a jump at any arbitrary time $t \geq 0$, then W cannot depend on the third variable. Then, $W(t, x, i) \equiv \tilde{W}(t, x)$ for some differentiable function $\tilde{W}$. In this case, $\tilde{W}$ has to be a strict Lyapunov function for the zero-input subsystems $\dot{x}(t) = f_{\mathrm{sw}}(t, x(t), 0, 1)$ and $\dot{x}(t) = f_{\mathrm{sw}}(t, x(t), 0, 2)$. However, each of these subsystems is stable but not asymptotically. As a consequence, there is no hope in finding an iISS-Lyapunov function V for the family (37) having a simple dependency on the switching signals σ in $\mathcal{S}$. This causes the computation of an iISS-Lyapunov function to be extremely complicated.

4.3. iISS of the SQZ-source inverter

Next, we show that there exists $\hat{\chi} \in \mathcal{K}_\infty$ such that (37) is iISS with $\mathbf{w}$ as the input and $L^1_{\text{loc},\hat{\chi}}(\mathbb{R}^{1+l})$ as the set of admissible inputs, provided that the time-varying system (35) is ISS. First, we note that the family of systems

$$\dot{x}(t) = f_\sigma(t, x(t), u_0(t) + h(t, z(t))), \quad \sigma \in \mathcal{S}, \tag{40}$$

is ISS if one considers $\text{col}[z, u_0, v]$ as input. In fact, since the parametrized family

$$\dot{x}(t) = f_\sigma(t, x(t), u(t)), \quad \sigma \in \mathcal{S}, \tag{41}$$

is ISS as explained above, there exist $\beta_1 \in \mathcal{KL}$ and $\rho_1 \in \mathcal{K}_\infty$ such that for every solution x of (41) corresponding to some $u \in L^\infty_{\text{loc}}(\mathbb{R}^m)$ and to some $\sigma \in \mathcal{S}$, and with initial time $t_0 \geq 0$, the estimate

$$|x(t)| \leq \beta_1(|x(t_0)|, t - t_0) + \rho_1(\|u_{[t_0,t]}\|) \quad \forall t \geq t_0$$

holds. Then, for every solution x of (40) corresponding to some $\text{col}[z, u_0, v] \in L^\infty_{\text{loc}}(\mathbb{R}^{k+1+l})$ and with initial time $t_0 \geq 0$, we have that for all $t \geq t_0$

$$\begin{aligned} |x(t)| &\leq \beta(|x(t_0)|, t - t_0) + \rho_1(\|(u_0 + \nu)_{[t_0,t]}\|) \\ &\leq \beta(|x(t_0)|, t - t_0) + \rho_1(\|u_{0[t_0,t]}\| + \|\nu_{[t_0,t]}\|), \end{aligned}$$

where we have defined $\nu(\cdot) = h(\cdot, z(\cdot))$ and used the triangle inequality. Since $h(t, 0) = 0$ for all $t \geq 0$ and h is locally Lipschitz in z, uniformly in $t \geq 0$, it follows that there exists a function $\hat{\rho} \in \mathcal{K}_\infty$ such that $|h(t, z)| \leq \hat{\rho}(|z|)$ for all $t \geq 0$ and all $z \in \mathbb{R}^k$. Then $\|\nu_{[t_0,t]}\| \leq \hat{\rho}(\|z_{[t_0,t]}\|)$ and

$$\begin{aligned} |x(t)| &\leq \beta(|x(t_0)|, t - t_0) + \rho_1(\|u_{0[t_0,t]}\| + \hat{\rho}(\|z_{[t_0,t]}\|)) \\ &\leq \beta(|x(t_0)|, t - t_0) + \tilde{\rho}_1(\|u_{0[t_0,t]}\|) + \tilde{\rho}_2(\|z_{[t_0,t]}\|) \end{aligned} \tag{42}$$

for all $t \geq t_0$. Here we have used the fact that $\rho_1(a+b) \leq \rho_1(2a) + \rho_1(2b)$ for all $a, b \geq 0$ and defined the class-$\mathcal{K}_\infty$ functions $\tilde{\rho}_1(s) = \rho_1(2s)$ and $\tilde{\rho}_2(s) = \rho_1(2\hat{\rho}(s))$. From (42) it easily follows that the family of systems (40) is ISS if one considers $\text{col}[z, u_0, v]$ as input. The system (35) is ISS with $\mathbf{w} = \text{col}[u_0, v]$ as input since it is ISS with v as input by assumption, and g does not depend on u_0. To assert that (37) is ISS, we employ the following generalization of the result for ISS of cascade time-invariant systems given in Proposition 3.2 of Jiang, Teel, and Praly (1994).

Proposition 4.1. *Consider the family of parametrized systems*

$$\begin{cases} \dot{x}(t) = g_1(t, x(t), z(t), w(t), \lambda) \\ \dot{z}(t) = g_2(t, z(t), w(t), \lambda), \end{cases} \quad \lambda \in \Lambda.$$

If the family of x-subsystems is ISS with $\text{col}[z, w]$ *as input and the family of z-subsystems is ISS with w as input, then the family of systems is ISS with w as input.*

Proof. Proposition 4.1 follows from the small-gain theorem given in Chen and Huang (2005, Thm. 2.1), by considering the set of constant functions $d : \mathbb{R}_{\geq 0} \to \Lambda$ as that of admissible disturbances. In the statement of Theorem 2.1 in Chen and Huang (2005), the disturbances $d(\cdot)$ take values in some Euclidean space $\mathbb{R}^{n_d}$ and the functions $f_1(t, x_1, v_1, u, d)$ and $f_2(t, x_2, v_2, u, d)$ are assumed piecewise continuous in t and d, and locally Lipschitz in (x_1, v_1, u) and (x_2, v_2, u), respectively. Nevertheless, the conclusions of such a theorem remain valid, with the same proof, if one assumes *constant* disturbances taking values in an arbitrary set Λ and functions f_1 and f_2 which are, respectively, Lebesgue measurable in t, locally Lipschitz in x_1 and x_2 and continuous in (v_1, u) and (v_2, u). ■

The existence of $\hat{\chi} \in \mathcal{K}_\infty$ such that the family (37) is iISS with $\text{col}[u_0, v]$ as input and $L^1_{\text{loc},\hat{\chi}}(\mathbb{R}^{1+l})$ as the set of admissible inputs then follows from Theorem 3.3, since the function $\mathbf{f}_\sigma$ satisfies $\mathbf{f}_\sigma(t, 0, 0) = 0$ for all $t \geq 0$ and all $\sigma \in \mathcal{S}$, and Assumption 1 with $\mathbf{x}$ in place of x and $\mathbf{w}$ in place of u.

5. Conclusion

We have provided a proof of the fact that ISS implies iISS, valid for parametrized families of time-varying systems. The proof does not employ Lyapunov characterizations of the corresponding stability properties, and hence is valid in settings where such characterizations do not exist. When particularized to time-invariant systems, the assumptions required are weaker than existing ones, since local Lipschitz continuity with respect to the input variable is not required. We have also shown that, for a time-varying system, local Lipschitz continuity in all variables is not sufficient to guarantee that ISS implies iISS. Our results also show how suitable iISS gains may be constructed based on the comparison functions that characterize the ISS property and on bounds on the function defining the system dynamics. Interesting questions that still remain open are whether ISS may imply iISS under non-uniqueness of solutions or for specific classes of infinite-dimensional systems.

Appendix A. Proof of Lemma 3.4

(B1)–(B2) ⇒ (A2)–(A3). That (A3) holds easily follows from (B1) by taking $L_2(r, s) := N(r) + O(s)$, $\omega := \tilde{\gamma}$ and noting that N and O are nondecreasing functions. It is also easy to see that (A2) follows from (B2) and Remark 3.5 by considering $L_1(r, s) := L(r)[P(r) + \gamma(s)]$.

(A2)–(A3) ⇒ (B1)–(B2). We assume in the following, according to Remark 3.1, that L_1 and L_2 in (A2)–(A3) are nondecreasing in each variable and continuous functions.

For proving (B1), we define for r, s and δ in $\mathbb{R}_{\geq 0}$

$$g(r, s, \delta) := \sup_{\substack{t \geq 0, \lambda \in \Lambda \\ |\xi| \leq r, |\mu_1| \leq s \\ |\mu_1 - \mu_2| \leq \delta}} |f_\lambda(t, \xi, \mu_1) - f_\lambda(t, \xi, \mu_2)|$$

We have that $g(r, s, \delta) \leq L_2(r, s + \delta)\omega(\delta)$ for all $r, s, \delta \geq 0$, and also that

$$|f_\lambda(t, \xi, \mu_1) - f_\lambda(t, \xi, \mu_2)| \leq g(|\xi|, |\mu_1| \wedge |\mu_2|, |\mu_1 - \mu_2|)$$

for all $t \geq 0$, $\xi \in \mathbb{R}^n$, $\mu_1, \mu_2 \in \mathbb{R}^m$ and all $\lambda \in \Lambda$. Since the function L_2 is nondecreasing in each variable,

$$L_2(r, s + \delta) \leq [1 + L_2(r, 2r) + L_2(s, 2s)][1 + L_2(\delta, 2\delta)].$$

Defining $\tilde{\gamma}, N, O : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ via

$$\tilde{\gamma}(\delta) = [1 + L_2(\delta, 2\delta)]\omega(\delta), \quad N(r) = 1 + L_2(r, 2r)$$

and

$$O(s) = L_2(s, 2s),$$

it follows that $\tilde{\gamma} \in \mathcal{K}_\infty$, that N and O are continuous and nondecreasing, and that

$$|f_\lambda(t, \xi, \mu_1) - f_\lambda(t, \xi, \mu_2)| \leq \tilde{\gamma}(|\mu_1 - \mu_2|)\big[N(|\xi|) + O\big(|\mu_1| \wedge |\mu_2|\big)\big]$$

showing that (B1) holds. Analogously, to establish (B2) we define

$$h(r, s, \delta) := \sup_{\substack{t \geq 0, \lambda \in \Lambda \\ |\xi_1| \leq r, |\mu| \leq s \\ |\xi_1 - \xi_2| \leq \delta}} |f_\lambda(t, \xi_1, \mu) - f_\lambda(t, \xi_2, \mu)|.$$

We have $h(r, s, \delta) \leq L_1(r + \delta, s)\delta$ and

$$|f_\lambda(t, \xi_1, \mu) - f_\lambda(t, \xi_2, \mu)| \leq h(|\xi_1| \wedge |\xi_2|, |\mu|, |\xi_1 - \xi_2|).$$

Since $L_1(\cdot,\cdot)$ is nondecreasing in each variable, then

$$L_1(r+\delta,s) \le [1+L_1(2\delta,\delta)][1+L_1(2r,r)+L_1(2s,s)].$$

Defining $\eta,\gamma,P:\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$ via

$$\eta(\delta)=[1+L_1(2\delta,\delta)]\delta, \quad \gamma(s)=L_1(2s,s)-L_1(0,0)+s$$

and

$$P(r)=1+L_1(0,0)+L_1(2r,r),$$

it follows that $\eta,\gamma\in\mathcal{K}_\infty$ and P is continuous and nondecreasing. It is clear that η satisfies (15) and that (B2) is satisfied. ■

Appendix B. Proof of Lemma 3.6

Consider $\eta,\gamma,\tilde{\gamma},N,O$ and P as in (B1)–(B2) of Lemma 3.4, and arbitrary $t\ge 0$, $\xi\in\mathbb{R}^n$, $\mu\in\mathbb{R}^m$ and $\lambda\in\Gamma$. Taking into account the fact that $f_\lambda(t,0,0)=0$ and Lemma 3.4, we have that

$$\begin{aligned}|f_\lambda(t,\xi,\mu)| &= |f_\lambda(t,\xi,\mu)-f_\lambda(t,0,0)|\\ &\le |f_\lambda(t,\xi,\mu)-f_\lambda(t,\xi,0)|\\ &\quad + |f_\lambda(t,\xi,0)-f_\lambda(t,0,0)|\\ &\le [N(|\xi|)+O(0)]\tilde{\gamma}(|\mu|)+P(0)\eta(|\xi|)\\ &\le \hat{N}(|\xi|)[1+\tilde{\gamma}(|\mu|)] \end{aligned} \tag{B.1}$$

where we have defined $\hat{N}(s)=\max\{P(0)\eta(s),O(0)+N(s),1\}$ for all $s\ge 0$. Since $\hat{N}$ is nondecreasing, then item (i) follows.

Let $\chi\in\mathcal{K}_\infty$ be such that $\chi\ge\max\{\gamma,\tilde{\gamma}\}$. Let $u\in L^1_{\mathrm{loc},\chi}(\mathbb{R}^m)$ and $\lambda\in\Lambda$, and consider $f_{u,\lambda}(t,\xi)=f_\lambda(t,\xi,u(t))$. First, note that for any interval $I\subset\mathbb{R}_{\ge 0}$, then $u_I\in L^1_{\mathrm{loc},\chi}(\mathbb{R}^m)$ and hence item (a) of Definition 2.2 is satisfied. Next, from (A3) in Assumption 1, it follows that $f_\lambda(t,\xi,\mu)$ is continuous in μ. From the latter continuity and the fact that $u(\cdot)$ is Lebesgue measurable, jointly with (A1), it follows that $f_{u,\lambda}(t,\xi)$ is Lebesgue measurable in t. From (A2) it straightforwardly follows that $f_{u,\lambda}(t,\xi)$ is continuous in ξ. Since $\chi\ge\max\{\gamma,\tilde{\gamma}\}$ and $u\in L^1_{\mathrm{loc},\chi}(\mathbb{R}^m)$, then both $\gamma(|u(\cdot)|)$ and $\tilde{\gamma}(|u(\cdot)|)$ are locally integrable. Let $t_2\ge t_1\ge 0$ and $r\ge 0$. From (B.1) and (B2) in Lemma 3.4, we have that for all $t_1\le t\le t_2$ and all $\xi_1,\xi_2\in B_r^n$, $|f_{u,\lambda}(t,\xi_1)|\le k_1(t)$ and $|f_{u,\lambda}(t,\xi_1)-f_{u,\lambda}(t,\xi_2)|\le k_2(t)|\xi_1-\xi_2|$, with $k_1(t)=\hat{N}(r)[1+\tilde{\gamma}(|u(t)|)]$, and $k_2(t)=L(r)[P(r)+\gamma(|u(t)|)]$ and $L(\cdot)$ as in Remark 3.5. Therefore, $f_{u,\lambda}$ satisfies the standard Carathéodory conditions for existence and uniqueness of solutions of ordinary differential equations (see Theorem I.5.3 in Hale (1980)) and hence item (b) of Definition 2.2 is satisfied. Last, item (c) of Definition 2.2 follows because, since $\chi\in\mathcal{K}_\infty$, any locally essentially bounded input u makes $\chi(|u(\cdot)|)$ locally integrable, and hence $L^\infty_{\mathrm{loc}}(\mathbb{R}^m)\subset L^1_{\mathrm{loc},\chi}$. ■

Appendix C. Proof of Lemma 3.11

Define $\bar{g}_2:\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$ via

$$\bar{g}_2(s)=s+\sup_{0\le t\le s} g_2(t).$$

Note that $\bar{g}_2$ is continuous, strictly increasing and unbounded. Therefore, $\bar{g}_2$ has an inverse, defined over the interval $[g_2(0),\infty)$. Define $c:=\max\{1,g_2(0)\}$ and $a:=\bar{g}_2^{-1}(c)$. Let $\phi:\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$ be defined via

$$\phi(s)=\bar{g}_2(s+a)-\bar{g}_2(a)=\bar{g}_2(s+a)-c,$$

and note that $\phi\in\mathcal{K}_\infty$, and that

$$\phi(s)+c=\bar{g}_2(s+a)\ge\bar{g}_2(s).$$

Define $\bar{g}_1:[c,\infty)\to\mathbb{R}_{>0}$ as

$$\bar{g}_1(r)=\sup_{c\le t\le r} g_1(t),$$

and note that $\bar{g}_1$ is nondecreasing. Define $\kappa\in\mathcal{K}_\infty$ via

$$\kappa(r):=\begin{cases} r\bar{g}_1(c) & \text{if } 0\le r<c,\\ r\bar{g}_1(r) & \text{if } r\ge c.\end{cases}$$

We thus have, for $r\ge\phi(s)+c\ge c$,

$$\kappa(r)=r\bar{g}_1(r)\ge[\phi(s)+c]g_1(r)\ge\bar{g}_2(s)g_1(r)\ge g_2(s)g_1(r).$$

This concludes the proof. ■

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