An Alternative Approach to the State Observation Problem For Lipschitz Continuous Systems with Controls

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Abstract— **In the present work we propose an alternative approach to the state observation problem for Lipschitz continuous systems with open-loop controls. We construct an observer based on a concept of observability weaker than that of the instantaneous observability. The resulting observation algorithm is then applicable under mild assumptions about the considered dynamical system.**

I. INTRODUCTION

In the past twenty years, considerable attention has been paid to the design of observers for dynamical systems (see [1], [2] and references therein for details). Although Lie algebraic methods are by now currently employed [3], they are restricted to the class of systems for which there exists a suitable state-space transformation. Smoothness is in this case instrumental in order to obtain such transformation.

The different models that extend the Luenberger model to Lipschitz continuous systems are based in the rather restrictive assumption that the original system can be decomposed into a linear observable part and a nonlinear Lipschitz continuous one with Lipschitz constant small enough ([4]). Another approach developed for non-smooth systems, the so-called *optimization based observer* is particularly appealing. This approach relies on the minimization of a cost functional over a moving horizon (see e.g. [5]). As the idea is to store measurements from an (sliding) interval $[t - T_0, t]$, and to generate a state estimate so as to asymptotically match the predicted output with the measured one on the whole interval, this observer concept involves an infinite dimensional structure, that can at best be approximately realized at the implementation stage. In addition, the minimization process involves the use of derivatives of the output function and of the vector field that determines the dynamics of the system (even up to order two in the aim of assuring convergence of the method). This fact precludes the use of the method for Lipschitz continuous systems, except in case when non-smooth optimization is used, a method that poses a tough problem. In addition, the problem could be non-convex, and consequently global optimization techniques should be used.

A different observer design that avoids the minimization stage of the optimization based observer, was presented in [1] for autonomous uncontrolled systems and in [6] and [7] for controlled ones. The principal difficulty of this design is that one must preprocess the output of the system for each input and then construct a partial inverse. This fact not only precludes the use of the observer in closed-loop with a controller, but also implies heavy computational burden when the class of open-loop inputs is large.

Several other approaches for the nonlinear observation problem have been developed in the last decade (see e.g. [8] and [9] and references therein) but in one way or another none of them is suitable under the weak assumptions about the system that we consider here.

This paper takes a different, more system theoretically oriented perspective of the subject. Given that instantaneous observability cannot be established for the systems we deal with, since no Lie-derivative based notion is in this case possible, we pursue the idea of an observation process based on the weakest notion of observability: discard recurrently the assumed initial states since they are distinguishable from the true initial state. Based on this simple idea, we develop an algorithm that sequentially reduces the space of possible states that at each time have the same output as the real one. Since the estimation algorithm evolves in time as the real system does, at its final step (when all possible states, except a neighborhood as small as desired of the real one are eliminated) we obtain an arbitrarily good estimate. Neither convexity nor smoothness are prerequisites for our approach to work.

The paper unfolds as follows. In section 2 we introduce some notation and state the problem. In section 3 we present the observer and prove its convergence. Section 4 exhibits some simulations and in section 5 we present the conclusions.

II. NOTATION AND PROBLEM STATEMENT

Throughout, $\mathbb{R}$ and $\mathbb{N}_0$ denote the sets of real and non-negative integer numbers respectively. Given $A \subset \mathbb{R}^p, p \in \mathbb{N}$, $\mathcal{CO}(A)$ denotes the class of compact subsets of A, $\#(A)$ the cardinal of A, and 2^A the power set induced from A. For any $\xi \in \mathbb{R}^p$, we denote $\text{dist}(\xi, A) = \inf\{|\xi - x|, x \in A\}$, where $|\cdot|$ is the euclidean norm in $\mathbb{R}^p$, and for any $\gamma > 0$, $(A)_\gamma := \{x : \text{dist}(x, A) \leq \gamma\}$. We also denote by $< \cdot, \cdot >$ the usual inner product and for $\xi \in \mathbb{R}^n$ and ε a positive number, $B(\xi, \varepsilon)$ is the open ball in $\mathbb{R}^n$ centered at ξ and with radius ε. Given $E \in \mathcal{CO}(\mathbb{R}^p)$ and a locally Lipschitz function $g : \mathbb{R}^p \to \mathbb{R}^q$ we denote L_g^E and $\|g\|_E$ the Lipschitz constant and supremum norm of g on E respectively. When referring to a time dependent variable, *e.g.* $y(t_k)$, we shall denote it y_k for short.

From now on, for binary as well as unary operations, we make the following abuse of notation: they are allowed to

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act on sets. For instance, given two sets $A = \{a, b\}$ and $B = \{c, d\}$, $A \star B = \{a \star c, a \star d, b \star c, b \star d\}$ for any given binary operation $\star$, and given an unary operation $\eta(\cdot)$, $\eta(A) = \{\eta(a), \eta(b)\}$.
In this paper we shall consider systems described by:

$$\dot{x} = f(x, u), \qquad y = \quad h(x), \tag{1}$$

where, for $t \geq 0$, $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^p$ is the output, and the input $u : [0, \infty) \rightarrow \mathbb{R}^m$ belongs to the class $\mathcal{U}$ of locally bounded Lebesgue measurable functions. We will assume that $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is locally Lipschitz continuous in the first argument uniformly in the second and $h : \mathbb{R}^n \rightarrow \mathbb{R}^p$ is locally Lipschitz continuous. Given $x_0 \in \mathbb{R}^n$ we denote by $x(t, t_0, x_0, u(t))$ the (unique) solution of $\dot{x} = f(x, u)$ with initial condition $x(t_0, t_0, x_0, u(t_0)) = x_0$. For simplicity, we define $f_u(x) := f(x, u)$ and accordingly we denote $T_{f_u}^{t_0,t}(x_0) = x(t, t_0, x_0, u(t))$ for short and, when it is clear from the context, $x(t) = x(t, t_0, x_0, u(t))$ and $y(t) = h(x(t))$.

As previously stated, we intend to produce an estimate of the state $x(t)$ of the system (1), based on a notion of observability as weak as possible compatible with the rather general hypotheses about the system. With this aim, let us recall two definitions related to the observation problem:

Definition 1: [INDISTINGUISHABILITY] A pair of states $(x_0, x_0') \in \mathbb{R}^n \times \mathbb{R}^n$ is *indistinguishable* for the system (1) if $\forall u \in \mathcal{U} \quad \forall t \geq t_0,\ h(T_{f_u}^{t_0,t}(x_0)) = h(T_{f_u}^{t_0,t}(x_0'))$.

Definition 2: [OBSERVABILITY] The system (1) is *observable at* x_0 if no $x_0' \in \mathbb{R}^n$ exists such that the pair (x_0, x_0') is indistinguishable. The system is *observable* if it is observable at x_0, for every $x_0 \in \mathbb{R}^n$.

We say that the system (1) is *observable over* $\mathcal{D}$ if in Def. 2 we replace $\mathbb{R}^n$ by $\mathcal{D} \subset \mathbb{R}^n$.

This notion of observability is the weakest one that can be postulated for this kind of systems (see [10]).

Remark 2.1: As can be seen, the notion of *universal inputs* (those that do not render states indistinguishable) underlies that of observability. No singular input can exist if observability holds.

A. Basic Principle

Let us first introduce the following concept:

Definition 3: [SEARCH-SPACE] Given some fixed $t \geq t_0$, a subset D_t of the state space is a *search-space* if it includes the actual state $x(t)$ of the system (1).
The observer we propose is a dynamical system that starts off with a search-space D_{t_0} and evolves in such a way that the search-space diminishes to an arbitrarily small neighborhood of the actual system state. This reduction will be done by means of a method drawn, conceptually, from the concept of observability. Although in principle D_{t_0} could be any set that contains x_0, we take $D_{t_0} = h^{-1}(y(t_0))$[1]. From there on, the dynamics of the system will play a fundamental role

[1]Where $h^{-1}(y) := \{x \in \mathbb{R}^n : h(x) = y\}$ the pre-image of y by h.

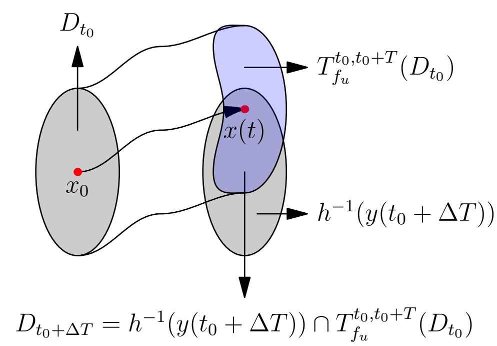

Fig. 1. Basic principle.

in determining the evolution of the search-space, as can be seen in Fig. 1. Let for some $\Delta T > 0$,

$$D_{t_0+\Delta T} = T_{f_u}^{t_0,t_0+\Delta T}(D_{t_0}) \cap h^{-1}(y(t_0+\Delta T))$$

Ideally $D_{t_0+\Delta T} \subsetneq T_{f_u}^{t_0,t_0+\Delta T}(D_{t_0})$ and only at worst $T_{f_u}^{t_0,t_0+\Delta T}(D_{t_0}) \subseteq h^{-1}(y(t_0+\Delta T))$. Due to observability, this should not occur for *every* ΔT since it would imply the existence of indistinguishable states. So, while the search space may not diminish for some discarding times ΔT, that should not always happen.

Remark 2.2: If the process of reduction of the search-space were to evolve in continuous time, observability would be enough for the observer to converge. In fact, suppose that there is a state $\xi \in D_{t_0}$ such that $\xi \neq x_0$. Then, if $T_{f_u}^{t_0,t}(\xi)$ survives the process of elimination for all $t > t_0$ the observer will not converge, but the system is not observable at x_0 either, because (ξ, x_0) is an indistinguishable pair by definition.

Remark 2.3: If we apply the observation mechanism described above to an observable linear system with scalar output, we can estimate the actual state of the observed system in at most $n-1$ steps, since we begin with a hyperplane of dimension $n-1$ and at each successive step we reduce the dimension of the original hyperplane (search-space) at least by one, therefore achieving a zero dimensional search-space, which contains the only state we are interested in.

Of course, in real life settings we will have to take into account several restrictions. In fact, we observe at discrete-time and with finite resolution both in the state space and in the output space, and in addition the search-space $h^{-1}(y_0)$ may not be bounded. In the sequel we suppose that the following holds:

Assumption 1. The search-space $D \in \mathcal{CO}(\mathbb{R}^n)$

In order to take into account the finite resolution in the observation of the output values, we introduce next alternative notions of indistinguishability and observability.

Definition 4: Given $\gamma > 0$ and $u \in \mathcal{U}$,

- A pair of states (x_1, x_2) is *γ-indistinguishable (wrt u)* for the system (1) if $\forall t \geq t_0$, $|h(T_{f_u}^{t_0,t}(x_1)) - h(T_{f_u}^{t_0,t}(x_2))| < \gamma$.
- The system (1) is *γ-observable (wrt u)* if for all $x_1, x_2 \in R^n$ with $x_1 \neq x_2$, there exists $t^* > t_0$ such that $|h(T_{f_u}^{t_0,t^*}(x_1)) - h(T_{f_u}^{t_0,t^*}(x_2))| > 2\gamma$.

• Given $D \subset \mathbb{R}^n$, system (1) is *γ-observable (wrt u) over D* if for all $x_1, x_2 \in D$ with $x_1 \neq x_2$, there exists $t^* > t_0$ such that $|h(T_{f_u}^{t_0,t^*}(x_1)) - h(T_{f_u}^{t_0,t^*}(x_2))| > 2\gamma$

Remark 2.4: The existence of such t^* states that any two different initial states in $\mathbb{R}^n$ may be distinguished in finite time from the outputs corresponding to the trajectories that for the given $u \in \mathcal{U}$ they generate, even when the resolution in the output is of order 2γ.

Remark 2.5: In the sequel we will assume that a fixed but otherwise arbitrary input $u \in \mathcal{U}$ is applied to system (1) and that all the input-dependent properties already introduced are referred to this input, and we will omit any reference to it.

The next result will be instrumental in what follows.

Proposition 2.1: Let γ, ε positive real numbers and suppose system (1) is γ-observable as in Definition 4. Let $E \in \mathcal{CO}(\mathbb{R}^n)$ and $x_0 \in E$. Then if $E_\varepsilon = \{\xi \in E : |\xi - x_0| \geq \varepsilon\} \neq \emptyset$, there exists $t^\star > 0$ that depends on E, x_0, u and ε such that for all $\xi \in E_\varepsilon$, $|h(T_{f_u}^{t_0,t}(\xi)) - h(T_{f_u}^{t_0,t}(x_0))| > 2\gamma$ for some $t \in [t_0, t^\star]$.

Proof:

Let $\mu(\xi, t) := |h(T_{f_u}^{t_0,t}(\xi)) - h(T_{f_u}^{t_0,t}(x_0))|$ and let $\xi \in E_\varepsilon$. Due to the γ-observability, there exists t_ξ such that $\mu(\xi, t_\xi) > 2\gamma$. Since $\mu(\cdot,\cdot) \in \mathcal{C}(E \times [t_0, \infty))$, there exists $0 < \delta_\xi < \varepsilon$ such that $\mu(\eta, t_\xi) > 2\gamma$ for every $\eta \in B(\xi, \delta_\xi)$. Since $\{B(\xi, \delta_\xi), \xi \in E_\varepsilon\}$ is an open covering of E_ε which is a compact set, there exists $\{\xi_1, \ldots, \xi_k\} \subset E_\varepsilon$ such that $E_\varepsilon \subset \cup_{i=1}^k B(\xi_i, \delta_{\xi_i})$. It follows readily that $t^\star = \max\{t_{\xi_1}, \ldots, t_{\xi_k}\}$ verifies the thesis. ■

We have already seen the basic principles and developed the intuition of the approach. In the next section we present the observer.

III. THE OBSERVER

We will first present a system, that we denote $\epsilon - \gamma$-**Observer**, that is the building block of the observer we propose, but previously, let us introduce two definitions:

Definition 5: Let $\epsilon > 0$,

• Given $\mathcal{C} \subset \mathbb{R}^n$, $R^\epsilon(\mathcal{C})$ is the set consisting of hypercubes of radius[2] ϵ whose centers are the elements of $\mathcal{C}$.

• $\mathcal{F}_\epsilon : \mathcal{CO}(\mathbb{R}^n) \to 2^{\mathbb{R}^n}$ is such that for any $D \in \mathcal{CO}(\mathbb{R}^n)$, $\mathcal{C} = \mathcal{F}_\epsilon(D)$ is the minimum finite set of points such that $D \subset \cup_{r \in R^\epsilon(\mathcal{C})} r$. If D itself is finite, then $\mathcal{F}_\epsilon(D) = D$. [3]

• Given an hypercube $r^\star \in R^\epsilon(\mathcal{C})$, $C^\epsilon(r^\star)$ is the center of $r^\star$.

• Given a finite set $\mathcal{C} = \{c_1, c_2, \ldots, c_k\} \subset \mathbb{R}^n$, we define the *indicator function* $S_\mathcal{C} : 2^\mathcal{C} \to \{0,1\}^k$ as:

$$S_\mathcal{C}(\bar{\mathcal{C}})^{(i)} = \begin{cases} 1 & \text{if } c_i \in \bar{\mathcal{C}} \\ 0 & \text{otherwise.} \end{cases}$$

for every set $\bar{\mathcal{C}} \subseteq \mathcal{C}$ [4]. Its inverse is well defined and can be trivially constructed.

[2]Given a set A of elements of $\mathbb{R}^n$ its radius is $\text{diam}(A)/2$

[3]If D is not finite $\mathcal{F}_\epsilon(D)$ as defined above needs not to be unique. Therefore, we will assume that we apply always the same criterion of selection.

[4]For example, $S_{\{c_1,c_2,c_3\}}(\{c_2\}) = (0, 1, 0)$.

Now we introduce the building block of the observer:

Definition 6: Given γ, ϵ positive numbers, an $\epsilon - \gamma$-**Observer** is a *system* $\mathcal{O}_{\epsilon,\gamma}^{\bar{\tau}}(D) = (\mathcal{T}, \mathcal{X}, \mathcal{Y}, \phi_\gamma)$ where:

• D is the *initial* search-space.

• The *time space* $\mathcal{T} = \{t_0 = \bar{\tau}, t_1, \ldots, t_k, \ldots\}$, is a given strictly increasing sequence of times.

• $\mathcal{X} = \{0,1\}^{\#(\mathcal{F}_\epsilon(D))}$ is the *state space*[5] and $\xi_0 = S_{\mathcal{F}_\epsilon(D)}(\mathcal{F}_\epsilon(D)) \in \mathcal{X}$ is the initial state.

• $\mathcal{Y}$ is the *input-value space* that coincides with the output-value space of the observed system.

• The *transition map* $\phi_\gamma : \mathcal{D}_{\phi_\gamma} \to \mathcal{X}$ with domain

$$\begin{aligned} \mathcal{D}_{\phi_\gamma} \subseteq \quad & \{(\tau, \sigma, \xi, \iota) : \sigma, \tau \in \mathcal{T}, \\ & \sigma \leq \tau, \xi \in \mathcal{X}, \iota \in \mathcal{Y}^{(\sigma,\tau]}\}, \end{aligned}$$

is defined by the recursion $\phi_\gamma(t_k, t_{k-1}, \xi_{k-1}, \{y_k\}) = \xi_k$ with

$$\xi_k^{(i)} := (|y_k - h(\bar{x})| > \gamma) \quad \forall i$$

where $\bar{x} = T_{f_u}^{t_{k-1},t_k}\left[S_{\mathcal{F}_\epsilon(D)}^{-1}(\bar{\xi})\right]$ with $\bar{\xi}^{(j)} = 0$ if $j \neq i$ and $\bar{\xi}^{(j)} = \xi_{k-1}^{(i)}$ if $j = i$, and

$$(a > b) := \begin{cases} 0 \text{ if } a > b \\ 1 \text{ otherwise.} \end{cases}$$

Observation 1: Suppose, for instance, that $x_0 = 0.5$ and $D = [-1, 1]$. If $\epsilon = 0.5$ then $\mathcal{F}_\epsilon(D)$ is $\{-1, -0.5, 0, 0.5, 1\}$ and $\xi_0 = (1, 1, 1, 1, 1)$. Eventually, the observer state will evolve towards $\xi = (0, 0, 0, 1, 0)$.

Remark 3.1: It is worth noting that whenever $|y - h(x)| > \gamma$ holds, the observer discards x. This is equivalent to taking intersections (see e.g. Fig. 1) and discarding the set $\{x \in D : T_{f_u}^{t_{k-1},t_k}(x) \notin (h^{-1}(y_k))_\gamma\}$, but no explicit computation of h^{-1} is necessary.

A. Observer Definition

We are now in position to define the Observer:

Definition 7: [OBSERVER] Given an initial search-space D the strictly increasing sequences $\mathcal{T} = \{t_k, k \in \mathbb{N}_0\}$, $\{\tau_i = t_{k_i}, i \in \mathbb{N}_0, k_0 = 0\} \subset \mathcal{T}$ and the decreasing sequences of positive numbers $\{\epsilon_i, i \in \mathbb{N}_0\}$,$\{\gamma_i, i \in \mathbb{N}_0\}$, the **Observer** is a system $\mathcal{O}(D) = (\mathcal{T}, \mathcal{X}, \mathcal{U}, \phi)$ given by an arbitrary concatenation of $\epsilon_i - \gamma_i$-Observers:

$$\mathcal{O} := \mathcal{O}_{\epsilon_0,\gamma_0}^{\tau_0}(D_0) \triangleright \mathcal{O}_{\epsilon_1,\gamma_1}^{\tau_1}(D_1) \triangleright \cdots \triangleright \mathcal{O}_{\epsilon_i,\gamma_i}^{\tau_i}(D_i) \triangleright \cdots$$

where the concatenation must be understood as the successive application (in the temporal sense) of $\epsilon - \gamma$-Observers, *i.e.*: given two systems $\mathcal{O}_{\epsilon_i,\gamma_i}^{\tau_i}(D_i) = (\mathcal{T}_i, \mathcal{X}_i, \mathcal{Y}, \phi_{\gamma_i})$, $\mathcal{O}_{\epsilon_{i+1},\gamma_{i+1}}^{\tau_{i+1}}(D_{i+1}) = (\mathcal{T}_{i+1}, \mathcal{X}_{i+1}, \mathcal{Y}, \phi_{\gamma_{i+1}})$, where $(\mathcal{T}_i = \{\tau_i, \ldots, t_{k_{i+1}-1}\}$, the concatenation: $\mathcal{O} = \mathcal{O}_{\epsilon_i,\gamma_i}^{\tau_i}(D_i) \triangleright \mathcal{O}_{\epsilon_{i+1},\gamma_{i+1}}^{\tau_{i+1}}(D_{i+1})$ is another system $\mathcal{O} = (\mathcal{T}, \mathcal{X}, \mathcal{Y}, \phi)$ where:

• $\mathcal{T} = \mathcal{T}_i \cup \mathcal{T}_{i+1}$,

• $\mathcal{X} = \mathcal{X}_i \times \mathcal{X}_{i+1}$,

• The transition map ϕ is given by[6]:

[5]Not to be confused with the state-space of the observed-system

[6]Note that for $\phi, \phi_{\gamma_i}, \phi_{\gamma_{i+1}}$ the state variable ξ belongs to different spaces; respectively, $\mathcal{X}, \mathcal{X}_i, \mathcal{X}_{i+1}$

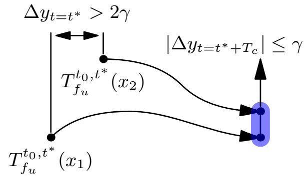

Fig. 2. State discarding.

$$\phi(t_k, \tau_i, \xi_{k-1}, \{y_0, \cdots, y_k\}) :=$$

$$\begin{cases} (\phi_{\gamma_i}(t_k, t_{k-1}, \xi_{k-1}^{\mathcal{X}_i}, \{y_k\}), (0, \ldots, 0)) & t_{k-1} \in \mathcal{T}_i \\ (\xi_{k-1}^{\mathcal{X}_i}, \quad \phi_{\gamma_{i+1}}(t_k, t_{k-1}, \xi_{k-1}^{\mathcal{X}_{i+1}}, \{y_k\})) & t_{k-1} \in \mathcal{T}_{i+1} \end{cases}$$

• $D_0 = \mathcal{F}_{\epsilon_0}(D)$ and for $i > 0$, D_i is defined recursively as follows: $D_i := [(X + G_{\epsilon_i}) \cup X] \cap \left[(h^{-1}(y(\tau_i)))_{\epsilon_i}\right]$, where the set $G_{\epsilon_i} = \{-\epsilon_i/\sqrt{n}, \epsilon_i/\sqrt{n}\}^n$ and $X = T_{f_u}^{\tau_{i-1},\tau_i}[S^{-1}_{\mathcal{F}_{\epsilon_{i-1}}(D_{i-1})}(\xi_{k_i}^{\mathcal{X}_{i-1}})]$.

Remark 3.2: Note that if Assumption 1. holds for the *initial* search-space, then D_i is finite for all $i \in \mathbb{N}_0$.

B. Convergence of the Observer

In order to assure the convergence of the observer $\mathcal{O}$, the following two conditions must hold:

• *Discarding*: given any $\epsilon - \gamma$-Observer belonging to the concatenation that defines $\mathcal{O}$, assuming that it is the only one in the concatenation, *i.e.* $\tau_1 = +\infty$, the sequence $a_k = |\phi(t_k, t_0, \xi_0, \{y_0, \ldots, y_k\})|$ is decreasing and has a sub-sequence that is *strictly* decreasing.

• *Consistency*: the hypercubes of the various $R^{\epsilon_i}(\mathcal{F}_{\epsilon_i}(D_i))$ that contain the actual state of the observed system should not be discarded, *i.e.* $\forall i \in \mathbb{N}_0$, given $r^* \in R^{\epsilon_i}(\mathcal{F}_{\epsilon_i}(D_i))$ such that $x(\tau_i) \in r^*$, then for all $t_k \in \mathcal{T}_i$:

$$\left\langle \phi(t_k, t_0, \xi_0, \{y_0, \ldots, y_k\})\}), S_{\mathcal{F}_{\epsilon_i}(D_i)}(C^{\epsilon_i}(r^*)) \right\rangle = 1$$

Let us first consider conditions for state discarding (i.e., search-space reduction) to hold. The state discarding problem is originated by the facts that in practice we only have a set of sampled outputs, with sampling times given by $\mathcal{T}$ and that we can determine the difference between two output values with a finite resolution. Figure 2 illustrates this problem: given $T_{f_u}^{t_0,t^*}(x_1)$ and $T_{f_u}^{t_0,t^*}(x_2)$ such that $|h(T_{f_u}^{t_0,t^*}(x_1)) - h(T_{f_u}^{t_0,t^*}(x_2))| > 2\gamma$, find the least value of T_c such that $|\Delta y| = |h(T_{f_u}^{t_0,t^*+T_c}(x_1)) - h(T_{f_u}^{t_0,t^*+T_c}(x_2))| \leq \gamma$, in other words, determine the least difference between sample times such that discarding is possible in case the discriminating condition occurs *between* them. The following result holds:

Lemma 1: [DISCARDING] Let γ, ϵ positive numbers and let the system (1) be γ-observable with search-space $D \subset E$ with $E \in \mathcal{CO}(\mathbb{R}^n)$ the ambient space for the trajectories of the system over the time interval of interest[7], and let the $\epsilon - \gamma$-Observer $\mathcal{O}^{\bar{\tau}}_{\epsilon,\gamma}(D) = (\mathcal{T}, \mathcal{X}, \mathcal{Y}, \phi_\gamma)$ where $\mathcal{T} = \{t_k = t_0 + k\Delta T, k \in \mathbb{N}_0\}$. If $\Delta T > 0$ verifies:

$$\Delta T < \frac{\gamma}{2L_h^E \|f_u\|_E}$$

then the discarding condition follows.

Proof:

Let k be such that $t_{k-1} \leq t^* < t_k$ and let $\bar{x} \in D$. It follows that $t_k \leq t^* + \Delta T$ and

$$|\bar{x} - T_{f_u}^{t^*,t_k}(\bar{x})| = \left| \int_{t^*}^{t_k} f(x(\tau, t^*, \bar{x}, u(t)))d\tau \right| \leq$$

$$\leq \int_{t^*}^{t^*+\Delta T} |f(x(\tau, t^*, \bar{x}, u(t)))|d\tau \leq \|f_u\|_E \cdot \Delta T.$$

Then:

$$\begin{aligned} |h(\bar{x}) - h(T_{f_u}^{t^*,t_k}(\bar{x}))| &\leq L_h^E |\bar{x} - T_{f_u}^{t^*,t_k}(\bar{x})| \\ &\leq L_h^E \|f_u\|_E \cdot \Delta T < \gamma/2 \end{aligned}$$

Now let x_1, x_2 in D, such that if $y_j(t) = h(T_{f_u}^{t_0,t}(x_j))$, $j = 1, 2$, $|y_1(t^*) - y_2(t^*)| \geq 2\gamma$. Then, since

$$\begin{aligned} 2\gamma \leq |y_1(t^*) - y_2(t^*)| &\leq |y_1(t^*) - y_1(t_k)| \\ &+ |y_1(t_k) - y_2(t_k)| \\ &+ |y_2(t^*) - y_2(t_k)|, \end{aligned}$$

$$|y_1(t_k) - y_2(t_k)| > \gamma$$

So, for a time window of length ΔT we can distinguish between states by observing the output and, as a consequence, discard states.

Finally, let $r^* \in R^\epsilon(\mathcal{F}_\epsilon(D))$ an hypercube that does not contain the actual state of the observed system. Let $x^* \in r^*$ and $t^\star$ as in Proposition 2.1, then $|h(T_{f_u}^{t_0,t^*}(x^*)) - y(t^*)| > 2\gamma$ for some $t^* = t^*_{x^*} < t^\star$. It follows that since there exists $k \in N_0$ such that $t_{k-1} \leq t^* < t_k$ then $a_k < a_{k-1}$, where $a_k = |\phi_\gamma(t_k, t_0, \xi_0, \{y(t_0), \ldots, y(t_k)\})|$.

■

Lemma 1 assures that for a fixed $\epsilon - \gamma$-Observer, state discarding will occur and, as a consequence, the search-space will be reduced arbitrarily for each $\epsilon > 0$. Anyway, there exists the possibility of discarding the whole search-space ($a_n \searrow 0$).

In order to avoid this possibility, we construct the observer as a concatenation of $\epsilon - \gamma$-Observers in such a way that *consistency* is guaranteed. We must first determine how long can a $\epsilon - \gamma$-Observer of the sequence be "active" without losing consistency.

Lemma 2: [CONSISTENCY] Given $\epsilon > 0$, $\gamma > 0$, and the corresponding $\epsilon - \gamma$-Observer, with D and E as in Lemma 1, if $\bar{t} \in \mathcal{T}$ verifies:

$$\bar{t} - t_0 < \frac{1}{L_{f_u}^E} \log\left(\frac{\gamma}{\epsilon L_h^E}\right)$$

then

$$\left\langle \phi_\gamma(\bar{t}, t_0, \xi_0, \{y(t_0), \ldots, y(\bar{t})\}), S_{\mathcal{F}_\epsilon(D)}(C^\epsilon(r^*)) \right\rangle = 1$$

where $r^* \in R^\epsilon(\mathcal{F}_\epsilon(D))$ contains the actual state of the observed system.

[7] it follows from Proposition 2.1 due to the γ-observability that this interval is finite.

Proof:

Let $c^* = C^\epsilon(r^*)$, the center of r^*. Then, for all $\eta \in r^*$,

$$|T_{f_u}^{t_0,\bar{t}}(c^*) - T_{f_u}^{t_0,\bar{t}}(\eta)| \leq \epsilon \cdot e^{L_{f_u}^E(\bar{t}-t_0)} \tag{2}$$

and r^* will not be discarded from the search-space (by the rule that defines ϕ_γ) as long as

$$|h(T_{f_u}^{t_0,\bar{t}}(c^*)) - h(x(\bar{t}))| \leq \gamma$$

From the Lipschitz condition of h and (2) we deduce:

$$\begin{aligned}|h(T_{f_u}^{t_0,\bar{t}}(c^*)) - h(x(\bar{t}))| &\leq L_h^E |T_{f_u}^{t_0,\bar{t}}(c^*) - x(\bar{t})| \\ &\leq L_h^E \cdot \epsilon \cdot e^{L_f^E(\bar{t}-t_0)} < \gamma.\end{aligned}$$

In consequence as long as $t \leq \bar{t}$, r^* will not be discarded by the $\epsilon - \gamma$-Observer, and consistency holds. ∎

Remark 3.3: At first glance, it seems that we can not have discarding and consistency at the same time. That is because discarding follows from the γ-observability, meanwhile consistency depends on the fact that the hypercube containing the actual system state is γ-indistinguishable. There is no contradiction because the latter must hold only for a certain finite period of time (as assured by the consistency lemma).

Next we present the main result of this paper.

Theorem 1: [Observer's Convergence] Suppose that the initial state of system (1) $x_0 \in D$, with $D \subset E$ the initial search-space, and E a sufficiently large ambient space where the system may evolve. Let $L_h^E < \kappa < 2L_h^E$, $\Delta T = \frac{\alpha\gamma_\star}{2L_h^E\|f_u\|_E}$ for some $\alpha \in (0,1)$ and some $\gamma_\star > 0$ and $\Delta\tau = K\Delta T$, then if:

1. $K = \max\left\{\lfloor \frac{2L_h^E\|f_u\|_E}{\gamma_\star L_f^E}\log\left(\frac{\kappa}{L_h^E}\right)\rfloor, 1\right\}$, where $\lfloor\cdot\rfloor$ is the integer part.
2. $\gamma_{i+1} = \max\{\frac{\gamma_i}{2}e^{L_f^E\Delta\tau}, \gamma_\star\}$ with $\gamma_0 \geq \gamma_\star$, $i \in \mathbb{N}_0$
3. $\gamma_i/\epsilon_i = \kappa$ for all $i \in \mathbb{N}_0$. If the system (1) is γ_i-observable over the sets $\Gamma(\epsilon_i)$[8], then for an observer $\mathcal{O}(D) = \mathcal{O}_{\epsilon_0,\gamma_0}^{t_0} \triangleright \mathcal{O}_{\epsilon_1,\gamma_1}^{t_0+\Delta\tau} \triangleright \cdots \triangleright \mathcal{O}_{\epsilon_i,\gamma_i}^{t_0+i\cdot\Delta\tau} \triangleright \cdots$ with $\mathcal{T} = \{t_0, t_0+\Delta T, \cdots, t_0+k\Delta T, \cdots\}$ the final state estimation error is less than or equal to $\epsilon_\star = \gamma_\star/\kappa$. Moreover, convergence is achieved in finite time.

Proof:

Notice that since $\kappa < 2L_h^E$ and due to hypothesis 2. and 3., $\{\gamma_i\}$ and $\{\epsilon_i\}$ are decreasing sequences with limits $\gamma_\star$ and $\epsilon_\star = \gamma_\star/\kappa$ respectively.

Consistency. Let $\mathcal{C} = \mathcal{F}_{\epsilon_0}(D)$, then, since $x_0 \in D$, it follows that there exists $c \in \mathcal{C}$ such that $x_0 \in R^{\epsilon_0}(\{c\})$. Given that $\Delta\tau_i = \Delta\tau = K\cdot\Delta T$ ($i \in \mathbb{N}_0$) and hypotheses 1. and 3., then:

$$0 < \Delta\tau_0 < \frac{1}{L_{f_u}^E}\log\left(\frac{\kappa}{L_h^E}\right) = \frac{1}{L_{f_u}^E}\log\left(\frac{\gamma_0}{\epsilon_0 L_h^E}\right),$$

and according to Lemma 2 c will not be discarded. Consider now the commutation from $\mathcal{O}_{\epsilon_0,\gamma_0}^{\tau_0}(D_0)$ to $\mathcal{O}_{\epsilon_1,\gamma_1}^{\tau_1}(D_1)$. Let $X = T_{f_u}^{\tau_0,\tau_1}[S_{\mathcal{F}_{\epsilon_0}(D_0)}^{-1}(\xi_{k_1}^{\mathcal{X}_0})]$, then by definition

$$D_1 = [(X + G_{\epsilon_1}) \cup X] \cap (h^{-1}(y(\tau_1)))_{\epsilon_1}.$$

[8] $\Gamma(\epsilon) := \{A \subset \mathbb{R}^n \times \mathbb{R}^n : \forall (x_1,x_2) \in A \quad |x_1 - x_2| \geq \epsilon\}$

Let $\overleftrightarrow{c} = T_{f_u}^{\tau_0,\tau_1}(c)$, $\eta_0 = \epsilon_0 e^{L_f^E\Delta\tau_0}$ and $\eta_1 = \eta_0/2$. Since

$$T_{f_u}^{\tau_0,\tau_1}(x_0) \in h^{-1}(y(\tau_1)) \cap R^{\eta_0}(\overleftrightarrow{c}) \text{ and}$$
$$R^{\eta_0}(\overleftrightarrow{c}) \subset R^{\eta_1}(\overleftrightarrow{c} + C_{\eta_1}),$$

then, there exists $\tilde{c} \in \{\overleftrightarrow{c} + C_{\eta_1}\}$ such that $T_{f_u}^{\tau_0,\tau_1}(x_0) \in R^{\epsilon_1}(\{\tilde{c}\})$ since by hypothesis $\epsilon_1 = \eta_1$ and, as $T_{f_u}^{\tau_0,\tau_1}(x_0) \in h^{-1}(y(\tau_1))$, $\mathrm{dist}(\tilde{c}, h^{-1}(y(\tau_1))) \leq \epsilon_1$. Then $\tilde{c} \in D_1$ and this state, that represents the hypercube where the system state lays at time τ_1, is not discarded in the process of obtaining D_1. From τ_1 on, consistency will follow from hypothesis 2. Consistency along the whole time scale follows by induction.

Discarding. Since ΔT verifies the hypothesis of Lemma 1, then for each $\epsilon - \gamma$-Observer of the concatenation discarding is warranted. It remains to prove that discarding is not affected by $\epsilon - \gamma$-Observer commutation. Suppose that at time τ_{i+1}, $|\xi_{k_{i+1}}^{\mathcal{X}_i}| > 1$ meaning that besides the center of the hypercube containing the real state of the system (and that will never be discarded by consistency), there is at least another element $c \in S_{\mathcal{F}_{\epsilon_i}(D_i)}^{-1}(\xi_{k_{i+1}}^{\mathcal{X}_i})$ that has not been discarded yet. By γ_i-observability over D_i (it can be easily seen that $D_i \subset \Gamma(\epsilon_i)$) and Lemma 1, if we do not commute from observer $\mathcal{O}_{\epsilon_i,\gamma_i}^{\tau_i}(D_i)$ to observer $\mathcal{O}_{\epsilon_{i+1},\gamma_{i+1}}^{\tau_{i+1}}(D_{i+1})$ then c will be discarded at some instant $t^* > \tau_{i+1}$, with $t^* \leq t^\star$ where $t^\star$ is given by Proposition 2.1 with $\gamma = \gamma_0$ and $\epsilon = \epsilon_\star$. On the other hand, as we actually commute $\epsilon - \gamma$-Observers, if not discarded by construction of D_{i+1} (or $D_{i+2}, \dots$), c will be discarded by the $\epsilon_j - \gamma_j$-Observer with j such that $\tau_j < t^* \leq \tau_{j+1}$ because it remains the same modulo the trajectory that passes trough it. As for the other elements of D_{i+1} that would be created as a consequence of c not being discarded, *i.e.* the set $\{T_{f_u}^{\tau_i,\tau_{i+1}}(c) + G_{\epsilon_{i+1}}\}$, they are all points that lay inside a set whose representative (c) was to be discarded at t^*, so they will also be discarded at most at τ_{j+1}. Finally as $\epsilon_i \to \epsilon_\star$ by construction, then convergence follows. ∎

Remark 3.4: Given a domain $\mathcal{D}$ and a number $\gamma > 0$, if a system is γ-observable over that domain, it is also $\bar{\gamma}$-observable for any $\bar{\gamma} < \gamma$. In addition, if a system is γ-observable over some set $\mathcal{D}$ then it is so for any set $\bar{\mathcal{D}} \subset \mathcal{D}$. In consequence, asking γ-observability for a fixed set $\mathcal{D}$ and a fixed $\gamma > 0$, is more restrictive than the observability hypothesis in Theorem 1.

Observation 2: Any observable linear system is γ-observable over the sets $\Gamma(\gamma/\kappa)$ for some $\kappa > 0$. Let $\omega(t) := Ce^{A(t-t_0)}$; since (C, A) is an observable pair, whichever be $\tau > t_0$, $W(\tau, t_0) = \int_{t_0}^{\tau}\omega^T(t)\omega(t)dt$ is definite positive. It follows that there exists $t_0 < t < \tau$ such that $|y(t,t_0,\bar{x},u) - y(t,t_0,x_0,u)|^2 \geq \sigma(\tau,t_0)|\bar{x} - x_0|^2/(2(\tau - t_0))$, with $\sigma(\tau,t_0)$ the least singular value of $W(\tau,t_0)$. It follows that if $\kappa \leq \sqrt{\sigma(\tau,t_0)/(8(\tau-t_0))}$ the system is γ-observable over the sets $\Gamma(\gamma/\kappa)$.

IV. An Example

With the purpose of exhibiting how the observer approach herein presented works, we consider the following Lipschitz continuous system:

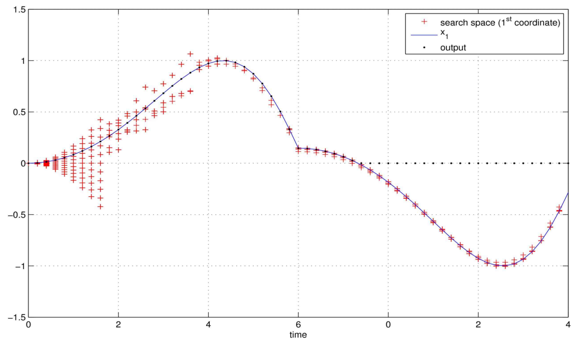

Fig. 3. search-space and output versus x_1 versus time.

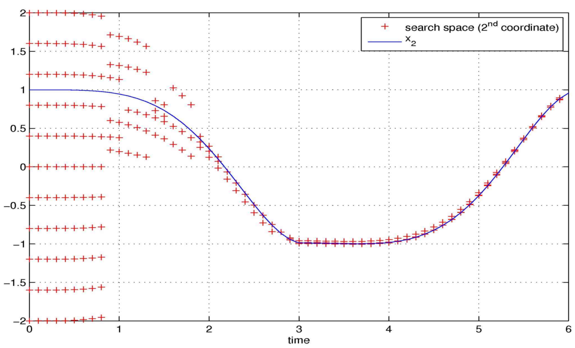

Fig. 4. search-space versus x_2 versus time.

$$\begin{cases} \dot{x_1} &= ux_2 \\ \dot{x_2} &= -ux_1 \\ y &= x_1 \cdot (x_1 > 0) \end{cases}$$

with input $u(t) = t/3 - \text{floor}(t/3)$ a sawtooth signal. It is clear that $f(x, u)$ is locally Lipschitz continuous uniformly on $\mathcal{U}$ with Lipschitz constant $L_f = \sup_{t\in[t_0,\infty)}\{|u(t)|\}$ ($L_f = 1$ in this case). $h(\cdot)$ has Lipschitz constant equal to 1. Observer parameters ΔT and $\Delta\tau$ are chosen small enough according to conditions imposed by the Theorem 1 and γ_0 is large enough so as to reduce computational burden at the first iterations for which the search-space is large. The evolution of the observer's dynamics can be seen in Figs. 3-4: the observer starts up with a large enough search-space (a $[-2, 2]$ segment embedded in $h^{-1}(y(t_0))$) that includes the initial condition ($x_0 = [0, 1]$) of the observed system. This set reduces its size as time evolves, rapidly reducing the algorithm's computing effort. As can be seen from Fig. 5, the estimation error[9] is reduced rapidly, in particular at $\epsilon - \gamma$-observer's commutations where the resolution is increased and further discarding follows. Finally, notice that the estimation error for x_1 is initially zero since $h^{-1}(y(t_0)) = \{(x_1, x_2) : x_2 \in \mathbb{R}, x_1 = y(t_0)\}$.

[9]For each $t \in \mathcal{T}$ the estimation error is defined as the maximum of the distances between the elements of the search-space and the current state trajectory (coordinate-wise).

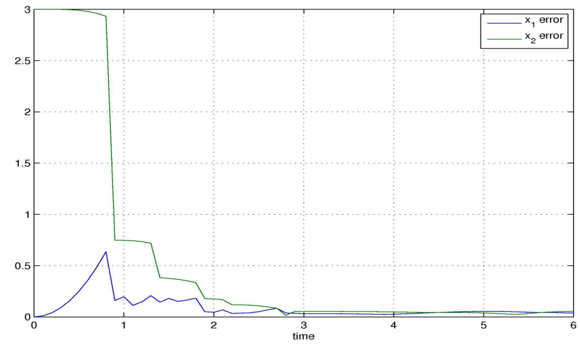

Fig. 5. Estimation error versus time.

V. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

We have proposed an observer that converges under rather nonrestrictive assumptions. This was achieved at the expense of an observer approach that is "parallel" in the sense that various possible states are evaluated at the same time. Nevertheless, such parallel structure is far from being brute-force, and therefore readily applicable in modern computers.

B. Future Works

Although we have presented an observer for autonomous controlled systems, it can be readily extended to the case of non-autonomous systems with open-loop controls. The case with closed-loop controls is more subtle and will be presented elsewhere.

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