# Chapter 24
# Quasi-analytical Perturbation Analysis of the Generalized Nonlinear Schrödinger Equation

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**Abstract.** The Generalized Nonlinear Schrödinger Equation (GNLSE) finds several applications, especially in describing pulse propagation in nonlinear fiber optics. A well-known and thoroughly studied phenomenon in nonlinear wave propagation is that of modulation instability (MI). MI is approached as a weak perturbation to a pump and the analysis is based on preserving those terms linear on the perturbation and disregarding higher-order terms. In this sense, the linear MI analysis is relevant to the understanding of the onset of many other nonlinear phenomena, but its application is limited to the evolution of the perturbation over short distances. In this work, we propose quasi-analytical approximations to the propagation of a perturbation consisting of additive white noise that go beyond the linear modulation instability analysis. Moreover, we show these approximations to be in excellent agreement with numerical simulations and experimental measurements.

## 24.1 Introduction

Pulse propagation in single-mode lossless nonlinear fibers is modeled by the Generalized Nonlinear Schrödinger Equation [1]

$$\frac{\partial A}{\partial z} - i\hat{\beta}A = i\hat{\gamma}A\int_{-\infty}^{\infty} R(T')\left|A(z,T-T')\right|^2 dT'. \qquad (24.1)$$

$A(z,T)$ is the pulse envelope, $z$ is the direction of propagation and $T$ is the time referred to a co-moving frame with group velocity $v_g = \beta_1^{-1}$ (i.e., $T = t - z\beta_1$). Linear dispersion is modeled by the operator $\hat{\beta}$, while $\hat{\gamma}$ is related to the third-order susceptibility:

$$\hat{\beta} = \sum_{k\geq 2} \frac{i^k \beta_k}{k!} \frac{\partial^k}{\partial T^k}, \; \hat{\gamma} = \sum_{k\geq 0} \frac{i^k \gamma_k}{k!} \frac{\partial^k}{\partial T^k}. \tag{24.2}$$

Finally, $R(T)$ models instantaneous and molecular Raman responses.

Analytical solutions of Eq. (24.1) are known in a variety of simplified cases. For example, solitonic solutions can be found by means of the inverse-scattering method originally proposed by Zakharov and Shabat [2] (see also, e.g., [3]), but only under some simplifying assumptions such as neglecting higher-order dispersion ($\beta_k = 0$ for $k \geq 3$). An important family of periodic solutions, known as Akhmediev breathers [4], has attracted attention in relation to supercontinuum generation and rogue waves [5,6]. Although Akhmediev breathers were originally found for low-order dispersion cases, Eq. (24.1) has been found to be integrable in more complex cases (see, for example, [7–11] and references therein). However, the number of exactly integrable variations of the GNLSE is still very limited.

Although exact solutions of simplified versions of Eq. (24.1) provide important insight on many characteristics of the propagation of pulses in nonlinear fibers, they cannot give a precise description in general. For this reason, the GNLSE is usually studied by means of simulations based on efficient algorithms such as split-step Fourier (SSF) [1] or a fourth-order Runge–Kutta in the interaction picture (RK4IP) [12].

In this work, we propose analytical approximations to the solution of Eq. (24.1) that provide a precise description of pulse propagation for a particular case of great interest. Our analysis focuses on a continuous-wave (CW) laser pumping the fiber. This CW pump is always accompanied by technical and quantum noise. One possibility is to approach noise propagation as a perturbation of the CW state. First-order perturbation or linear stability analysis is related to the study of the modulation instability (MI) phenomenon [4,5,13–23,23–29] (see also Chapter 5 of Ref. [1] and references therein). Exact solutions of MI accounting for the complete GNLSE have also been developed [30,31]. The particular case of the propagation of additive noise has been dealt with in the literature (see, e.g., [32,33]).

The wave propagation analysis of a noisy CW pump in an MI setting has several limitations. The continuous-wave pump is assumed undepleted and, hence, results are valid for short propagation distances. Furthermore, as it is a first order perturbation analysis, it disregards the four-wave mixing 'cascading effect', in the sense that perturbations to the pump, in turn, act as pumps themselves as soon as they attain enough power. One alternative to incorporate such cascading effect is to solve the GNLSE through Picard's iterations. Resulting expressions are, nevertheless, not easily tractable and even evaluating them numerically may be an expensive computational effort as compared to pure numerical solutions obtained from the usual SSF or RK4IP algorithms. For this reason, we put forth several simplifications that allow a simpler analysis of higher-order perturbations. The validity of these simplifications is tested through numerical and experimental studies.

It must be mentioned that there are alternative approaches which are related to ideas presented in this work. In particular, many tools have been developed for the statistical analysis of optical wave turbulence (see, e.g., [34–38]).

The remaining of this paper is organized as follows. In Sect. 24.2 we develop a higher-order perturbation analysis of the GNLSE and motivate the simplifications that allow tractability. We validate our approach with experiments and simulations in Sect. 24.3. Finally, conclusions are presented in Sect. 24.4.

## 24.2 Higher-Order Perturbation

Let us again consider the generalized nonlinear Schrödinger equation. It is useful to normalize the propagation distance as $\zeta = \gamma_0 P_0 z$. We study the propagation of a small perturbation $a(\zeta, T)$ to the stationary solution of Eq. (24.1), i.e., we consider $A(\zeta, T) = \sqrt{P_0}\,[1 + a(\zeta, T)]\, e^{i\zeta}$. Fourier transformation (with respect to time $T$) leads to

$$\frac{\partial \tilde{\mathbf{a}}(\zeta, \Omega)}{\partial \zeta} = \mathbf{A}(\Omega)\tilde{\mathbf{a}}(\zeta, \Omega) + \tilde{\mathbf{N}}(\tilde{\mathbf{a}}(\zeta, \Omega)), \tag{24.3}$$

where $\tilde{\mathbf{a}}(\zeta, \Omega) = \left[\tilde{a}(\zeta, \Omega), \overline{\tilde{a}(\zeta, -\Omega)}\right]^T$, with $\tilde{a}(\zeta, \Omega)$ the Fourier transform of $a(\zeta, T)$. The linear and nonlinear terms in the right-hand side are defined by

$$\mathbf{A} = i\begin{bmatrix} B(\Omega) & C(\Omega) \\ -B(-\Omega) & -\overline{C(-\Omega)} \end{bmatrix}, \; \tilde{\mathbf{N}}(\tilde{\mathbf{a}}(\zeta, \Omega)) = \begin{bmatrix} \tilde{\gamma}(\Omega)\tilde{N}\left(\tilde{a}(\zeta, \Omega)\right) \\ \tilde{\gamma}(-\Omega)\overline{\tilde{N}\left(\tilde{a}(\zeta, \Omega)\right)} \end{bmatrix}, \tag{24.4}$$

where $B(\Omega) = \tilde{\beta}(\Omega) + \tilde{\gamma}(\Omega)[1 + \tilde{R}(\Omega)] - 1$, $C(\Omega) = \tilde{\gamma}(\Omega)\tilde{R}(\Omega)$,

$$\tilde{\beta}(\Omega) = \frac{1}{\gamma_0 P_0}\sum_{m=2}^{M}\frac{(-1)^m}{m!}\beta_m \Omega^m, \qquad \tilde{\gamma}(\Omega) = \frac{1}{\gamma_0}\sum_{n=0}^{N}\frac{(-1)^n}{n!}\gamma_n \Omega^n, \tag{24.5}$$

$$\begin{aligned} \tilde{N}(\tilde{a}) = \tilde{R}(\Omega)\left[\tilde{a}(\zeta, \Omega) * \overline{\tilde{a}}(\zeta, -\Omega)\right] + \\ \tilde{a}(\zeta, \Omega) * \left[\tilde{R}(\Omega)\left(\tilde{a}(\zeta, \Omega) + \overline{\tilde{a}}(\zeta, -\Omega)\right)\right] + \\ \tilde{a}(\zeta, \Omega) * \left[\tilde{R}(\Omega)\left[\tilde{a}(\zeta, \Omega) * \overline{\tilde{a}}(\zeta, -\Omega)\right]\right], \end{aligned} \tag{24.6}$$

and $\tilde{R}(\Omega)$ is the Fourier transform of $R(T)$. For the sake of simplicity, in this work we let $\tilde{R}(\Omega) = 1$, that is, we neglect stimulated Raman scattering in the analysis.

Let us focus on the case where $a(0, T)$ is white noise. In particular, we assume that the mean power spectral density $s = \langle|\tilde{a}(0, \Omega)|^2\rangle$ is constant and that $\langle\tilde{a}(0, \Omega_1)\tilde{a}(0, \Omega_2)\rangle = 0$ and $\langle\tilde{a}(0, \Omega_1)\overline{\tilde{a}}(0, \Omega_2)\rangle = 0$ for $\Omega_1 \neq \Omega_2$. Using these hypotheses, it is simple to show [32, 33] that the solution to Eq. (24.3) when the nonlinear term is neglected is given by

$$\langle|\tilde{a}_0(\zeta,\Omega)|^2\rangle = \left\{\cosh\left(2G_1(\Omega)\zeta\right) - \frac{\left(\frac{B(\Omega)+B(-\Omega)}{2}\right)^2 - G_1^2(\Omega) + \tilde{\gamma}^2(\Omega)}{\left(\frac{B(\Omega)+B(-\Omega)}{2}\right)^2 + G_1^2(\Omega) + \tilde{\gamma}^2(\Omega)}\right\} \times \frac{\left(\frac{B(\Omega)+B(-\Omega)}{2}\right)^2 + G_1^2(\Omega) + \tilde{\gamma}^2(\Omega)}{2G_1^2(\Omega)} s, \tag{24.7}$$

where $G_1(\Omega)$ is the MI gain given by

$$G_1(\Omega) = \frac{\sqrt{4c(\Omega) - b^2(\Omega)}}{2}, \tag{24.8}$$

with $b(\Omega) = B(-\Omega) - B(\Omega)$ and $c(\Omega) = C(\Omega)C(-\Omega) - B(\Omega)B(-\Omega)$. Let us assume that there is gain, i.e., $G_1(\Omega) \in \mathbb{R}$, for some $\Omega$. Then, we may approximate

$$\langle|\tilde{a}_0(\zeta,\Omega)|^2\rangle \approx s + \left(e^{2G_1(\Omega)\zeta} - 1\right)|A_1(\Omega)|^2 s. \tag{24.9}$$

where

$$|A_1(\Omega)|^2 = \frac{\left(\frac{B(\Omega)+B(-\Omega)}{2}\right)^2 + G_1^2(\Omega) + \tilde{\gamma}^2(\Omega)}{2G_1^2(\Omega)}. \tag{24.10}$$

Equations (24.9)–(24.10) suggest the perturbative *ansatz*

$$\tilde{a}(\zeta,\Omega) \approx \sqrt{s}e^{i\phi_0(\zeta,\Omega)} + \sum_{n=1}^{\infty}\left(e^{G_n(\Omega)\zeta} - 1\right)A_n(\Omega)\sqrt{s^n}e^{i\phi_n(\zeta,\Omega)}. \tag{24.11}$$

Substitution of Eq. (24.11) in Eq. (24.3), along with the formal computation of the mean power spectral density, allows the determination of $A_n$ and $G_n$. Since the equations are quite involved, several simplifications must be made. One of the main simplifying assumptions is that $\langle\exp\{i(\phi_n(x,\mu) - \phi_m(y,\nu))\}\rangle = 0$ if either $n \neq m$, $x \neq y$ or $\mu \neq \nu$. After some tedious computations, it may be shown that, for $n \geq 2$

$$G_n(\Omega) \approx \max_{\mu}\left[G_1(\mu) + G_{n-1}(\Omega - \mu)\right], \tag{24.12}$$

$$|A_n(\Omega)| \approx \Delta_{\Omega}^{n-1} J(G_n(\Omega), \Omega), \tag{24.13}$$

where $\Delta_{\Omega}$ is a positive constant and

$$J(g,\Omega) = \frac{\sqrt{\left|\overline{B}(-\Omega) - ig\right|^2 |\tilde{\gamma}(\Omega)|^2 + \left|\overline{C}(-\Omega)\right|^2 \left|\overline{\tilde{\gamma}}(-\Omega)\right|^2}}{\left|[B(\Omega) + ig]\left[\overline{B}(-\Omega) - ig\right] - C(\Omega)\overline{C}(-\Omega)\right|}. \tag{24.14}$$

Although we do not present the details of the calculations due to the lack of space, some intuition on Eq. (24.12) may be gained by referring to the nonlinear

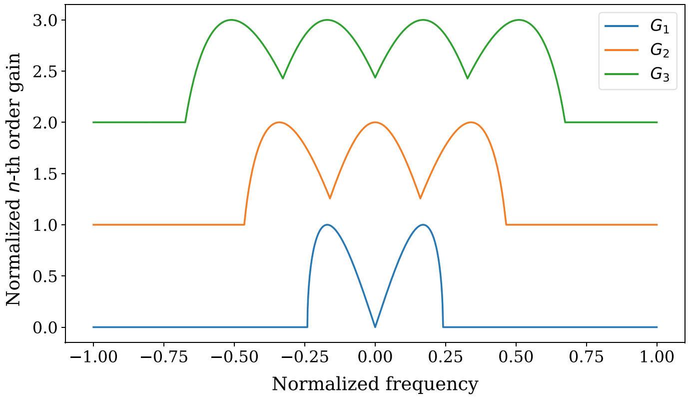


**Fig. 24.1.** Normalized gain for different perturbation orders. As the order increases, the gain captures the cascading effect of four-wave mixing

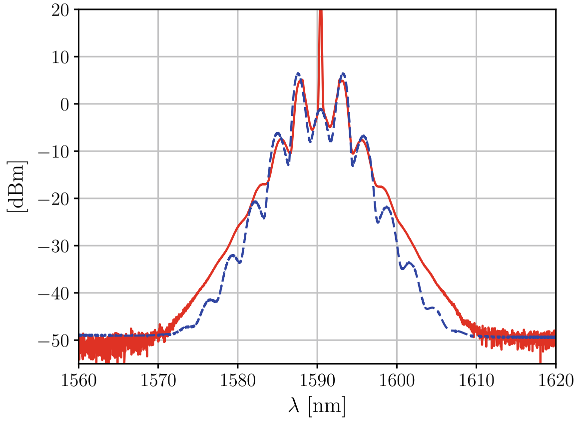


**Fig. 24.2.** Analytical approximation (blue dashed line) vs. experimental results (red solid line). A CW 30-dBm pump laser at 1590.4 nm was launched at the input end of the 770-m long dispersion-stabilized HNLF

operator in Eq. (24.6). The sum in Eq. (24.12) arises from the convolutions in the nonlinear operator. We are able to simplify the corresponding integrals by assuming that results are dominated by the largest gain and thus we take the maximum value. Figure 24.1 shows that, as the perturbation order $n$ increases, $G_n$ captures the cascading effect of four-wave mixing. Indeed, $G_1$ represents the well-known MI-gain due to the pump. $G_{n+1}$ incorporates the gain due to the perturbations amplified by $G_n$ acting as $n$th order 'pumps'.

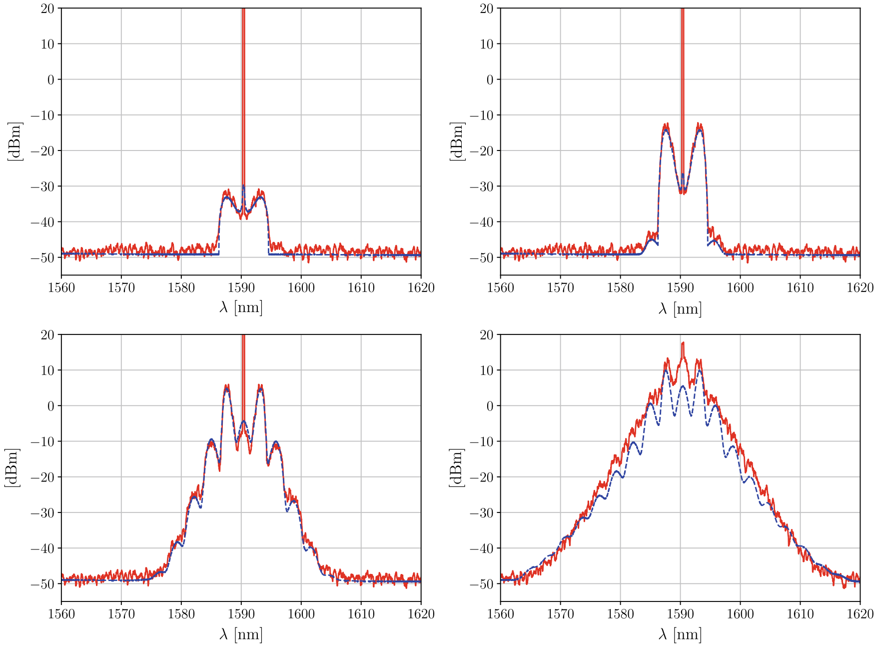


**Fig. 24.3.** Analytical approximation (blue dashed line) versus numerical results (red solid line) for different propagated distances: ~0.25 km (top left), ~0.50 km (top right), ~0.75 km (bottom left) and ~1 km (bottom right)

## 24.3 Experimental and Numerical Results

In order to test our approach we performed measurements of MI in a 770 m-long, dispersion-stabilized [39] Highly-Nonlinear Fiber (HNLF). A CW 30-dBm pump laser at 1590.4 nm was launched at the input end of the fiber. Figure 24.2 presents a comparison between the observed power (measured with 0.1-nm resolution) and the quasi-analytical approximation. The latter was obtained by using Eqs. (24.11)–(24.14) (adding up to $n = 8$) with $\gamma_0 = 8.7\ \mathrm{W}^{-1}\mathrm{Km}^{-1}$, $\gamma_k = 0$ for $k > 0$, $\beta_2 = -3.9198\ \mathrm{ps}^2/\mathrm{km}$, $\beta_3 = 0.1267\ \mathrm{ps}^3/\mathrm{km}$, $\beta_4 = 1.7594 \times 10^{-4}\ \mathrm{ps}^4/\mathrm{km}$ and $\beta_k = 0$ for $k > 4$. As it is readily observed, experimental, and analytical results are in excellent agreement.

In order to further explore the validity of the approximations, we performed computer simulations using the split-step Fourier algorithm. Figure 24.3 shows that the accuracy of the approximation decreases with the propagation distance, although reasonable good results are obtained even after 1 km. Figure 24.4 shows how approximations improve as the number of terms in Eq. (24.11) increases. Comparison to Fig. 24.1 helps to understand that the increasing detail is a consequence of the incorporation of the cascading four-wave mixing effect through higher-order perturbation gains $G_n$.

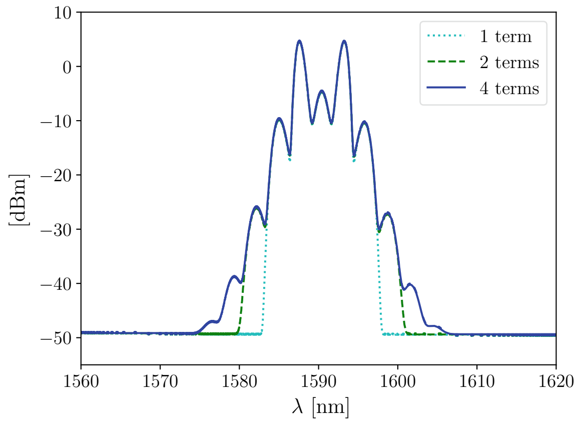


**Fig. 24.4.** Analytical approximation when increasing orders of approximation are used, at a propagation distance ∼0.75 km

## 24.4 Conclusions

A continuous-wave laser pump is always accompanied with technical and quantum noise. Thus, the propagation of a CW pump in a nonlinear optical fiber is a complex process. Its study is usually based on two different tools: numerical simulations and first-order linear stability (MI) analysis. While computer simulations are useful, they tend to hide the underlying basic physics. On the contrary, the modulation instability analysis gives some insights on the initial stages of propagation but fails at providing an accurate picture for longer propagated distances.

In this work, we put forth a perturbation analysis that offers both a precise description and meaningful physical insights. In particular, we showed our formulas to be accurate by comparing their predictions to actual experimental results. Furthermore, we validated our approximations with numerical simulations for propagated distances up to 1 km. The perturbation analysis also reveals the relevance of the cascading effect of four-wave mixing. In simple words, we might understand how produced MI gain spectra act as a new pumps further on.

The derivation of our approximation is complex and involves many simplifying assumptions. It is a matter of future work to look for a shorter path and less restrictive simplifications. It must be noted that, while those simplifications lead to extremely simple formulas, they may hide some interesting phenomena. For instance, it may be argued that the cascading effect of four-wave mixing is implicitly embedded in our choice of keeping only the largest gain in Eq. (24.12), but such an approximation might neglect relevant details appearing at longer distances (see Fig. 24.2). Finally, we believe our analysis to be of value when studying the early stages of supercontinuum generation and to contribute tools for the better understanding of rogue-wave formation.

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