A state estimation strategy for a nonlinear switched system with unknown switching signals

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ABSTRACT

A strategy is presented to estimate the state of a nonlinear autonomous switched system, with no knowledge of the switching signal, except its dwell time. To do so, algorithms to estimate the switching times and the current mode of the system are developed. The estimation of the switching times is based on approximating the second (generalised) derivative of the output of the system via a convolution of this signal with a suitable function and on detecting the corresponding spikes. To estimate the modes, a scheme based on the use of a bank of observers (one for each mode) and of a bank of subsystems (for each step of the estimation process a suitable subset of the subsystems of the switched system) is developed. The algorithms run regardless of the state observer model, as long as its output error norm decays exponentially with a controlled decay rate.

KEYWORDS

Switched systems; observers; switching times detection; mode detection

1. Introduction

A hybrid dynamical system consists of a family of continuous time systems (generally described by differential equations or differential inclusions) that switch among them according to a discrete rule, often modelled by a discrete event system. This class of systems is ubiquitous since it models the continuous and discrete interactions that appear in complex systems.

A switched system is a hybrid system whose switchings are performed by a switching signal and not by a discrete event system. Hence, a switched system is a finite numbers continuous-time dynamical subsystem and a switching rule (usually time- or state-dependent), called a switching signal, that determines the switching between the subsystems. In recent years, there has been an increasing interest in the control problems posed by switched systems due to their significance (see Decarlo et al., 2000; Liberzon, 2003; Liberzon & Morse, 1999; Lin & Antsaklis, 2009; Shorten et al., 2007 and references therein).

The state estimation problem has been investigated by researchers for many decades. For switched systems, this problem presents an interesting challenge, given the interaction between the discrete and the continuous dynamics present in this type of systems. The problem of designing observers for switched systems is acknowledged as an important topic of research (see Alesandri & Coletta, 2001; Petterson, 2006, among others). In this respect, the most demanding problem is that of simultaneously estimating the switching times, the continuous state and which subsystem (mode) governs the evolution of the continuous state. In the case in which the switching signal is only time dependent, it was shown that the observability of the state and that of the switching signal are independent properties (Gomez-Gutierrez et al., 2012) and hence suitable strategies of estimation must be developed for each of them.

For switched linear systems, i.e. those switched systems whose subsystems are linear, different strategies were developed to estimate the switching times (Laboudi et al., 2019; Tian et al., 2009), the mode (Lee et al., 2013), the continuous state (Alesandri & Coletta, 2001; Petterson, 2006; Tanwani et al., 2013; Xie & Wang, 2004) and both the mode and the continuous state (Rios et al., 2012; Vidal et al., 2003).

In the case of nonlinear switched systems, strategies based on sliding modes were used in, among others, Karami et al. (2019) to estimate the switching times, and in Davila et al. (2012) to estimate the continuous states and the mode. Designs based on Lyapunov techniques were presented in Ngoc Dinh and Defoort (2019) to estimate the continuous state and in Barhoumi et al. (2012) to estimate the mode (the continuous state was estimated in this case via high gain observers).

The aim of this paper is to develop an observation strategy for nonlinear switched systems in order to estimate the switching times, the modes and the continuous states of the system from the measurement of its output. It is only assumed that the switching signal is time dependent and that it has a well-defined dwell-time.

The technique to estimate the switching times is developed first for the case of a single input system. The method is based on approximating the second (generalised) derivative of the output of the system via a convolution of this signal with a suitable function and on detecting the corresponding spikes. To do so, a switching detectability condition, which only takes into account the first derivative of the output, is imposed. This technique is a posteriori generalised to the multiple output case.

In order to estimate the modes, a scheme based on the use of a bank of observers (one for each mode) and of a bank of subsystems (for each step of the estimation process a suitable

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subset of the subsystems of the switched system) is developed. The only requirement imposed on the observers is that the norm of the output error must decay exponentially, with a controlled decay rate. In this paper, we do not present a new model of the observer for each of the subsystems, but assume that it is already designed. In this way, since the focus is put on estimating the switching times and the current mode, great flexibility is obtained by selecting the model of the state observers among those available in the literature that match the aforementioned requirement.

This paper unfolds as follows. Section 2 introduces some notation, definitions and assumptions that are used throughout the work and states the problem. In Section 3, a method for detecting the switching times is proposed, first for the case of a single output and afterwords for the multiple outputs case. The estimator of the mode and that of the continuous state are presented in Section 4. In Section 5, a numerical example is added in order to illustrate the effectiveness of the proposed scheme. The high gain observer model used for each subsystem in the example was presented in Gauthier et al. (1992). Finally, in Section 6, the conclusions are presented.

2. Basic definitions and problem statement

In this work, we consider a nonlinear switched system

$$\begin{cases} \dot{x} = f_{\sigma(t)}(x(t)), \\ y = h(x(t)), \end{cases} \tag{1}$$

where $x \in \mathcal{U} \subset \mathbb{R}^n$ is the continuous state vector, with $\mathcal{U}$ an open set. $\sigma : \mathbb{R}_{\geq 0} \to \mathcal{Q}$, with $\mathcal{Q} = \{1, \dots, N\}$ the index set, is the switching signal, a piecewise constant and continuous from the right function. $f_q : \mathcal{U} \to \mathbb{R}^n, q \in \mathcal{Q}$, and $h : \mathcal{U} \to \mathbb{R}^p$ are sufficiently smooth vector fields and output map, respectively.

A (forward) solution of (1) corresponding to a switching signal σ is a locally absolutely continuous function $x : [t_0, t_f) \to \mathbb{R}^n$, with $0 \leq t_0 < t_f$, such that $\dot{x}(t) = f_{\sigma(t)}(x(t))$ for almost all $t \in [t_0, t_f)$.

Remark 2.1: In the sequel, we will assume that the state of system (1) evolves in a compact set, and hence $t_f = +\infty$.

The following notation will be used in the paper:

- Given a vector field η and a real-value function ω, both sufficiently smooth, $L_\eta^k \omega$, $k \in \mathbb{N}_0$ is the Lie derivative of ω of order k with respect to η.
- Let η as above and $\mu = (\mu_1, \dots, \mu_p)$ a smooth map. Then we denote $L_\eta^k \mu = (L_\eta^k \mu_1, \dots, L_\eta^k \mu_p)$.
- $\phi_q(t, t_0, x_0)$: is the solution of (1) defined for almost all $t \geq t_0$ such that $\sigma(\tau) = q$ for $t_0 \leq \tau \leq t$, and that $\phi_q(t_0, t_0, x_0) = x_0$.
- $\mathcal{Y}(t, t_0, x_0, q) = h(\phi_q(t, t_0, x_0))$.
- For any $x \in \mathbb{R}^s$, we denote with $\|x\|$ its Euclidean norm, and for any $A \in \mathbb{R}^{r\times s}$, $\|A\|$ its associated norm.
- With ∂A we denote the boundary of set A.
- Given $A \in \mathbb{R}^{r\times s}$, we denote by $\underline{\lambda}(A)$ and $\overline{\lambda}(A)$ the minimum and maximum singular values of A, respectively. Note that $\|A\| = \overline{\lambda}(A)$.
- Let a function $f : \mathbb{R} \to \mathbb{R}$ and $n \in \mathbb{N}$. With $D^{(n)}f$ we denote its nth generalised derivative.
- Given n a non-negative integer, $\delta^{(n)}(t)$ denotes the nth (generalised) derivative of the Dirac delta function $\delta(t)$.

Assumption 2.1: *The switching signal $\sigma(t)$ satisfies $t_{j+1} - t_j \geq \tau_D$ (dwell time), with $\tau_D > 0$ for all j, where t_j are the switching times. With this assumption, we avoid Zeno dynamics.*

In the following, it will be assumed that τ_D is large enough to determine the current mode of the system and to obtain a good estimation of the states.

The next notion refers to the property of the output of a system that enables to determine which mode is active at each time instant.

Definition 2.2:
- Two modes $q, q^\star \in \mathcal{Q}$, $q \neq q^\star$ of system (1) are *indistinguishable by the output*, if there exist initial states $x_0, x_0' \in \mathcal{U}$ such that $\mathcal{Y}(\tau, t_0, x_0, q) = \mathcal{Y}(\tau, t_0, x_0', q^\star)$, $\forall t_0 \leq \tau < t$, $\forall t < t_0 + \tau_D$.
- System (1) is *jointly observable* if there do not exist pairs of modes indistinguishable by the output.

We note that if system (1) fails to be jointly observable, a time interval might exist in which the current mode of the system could not be determined from the knowledge of its output.

Assumption 2.3: *System* (1) *is jointly observable.*

The next assumption, that is formulated in many papers concerning high gain observers (see, for instance, Ciccarella et al., 1993; Gauthier et al., 1992), will be instrumental in the sequel.

Assumption 2.4: *The state of the system takes values in a compact convex set $\mathcal{X} \subset \mathcal{U}$.*

The problem we address in this paper may be stated as follows.

Given the measurements of the output $y(t) = h(x(t))$ of system (1):

- *determine the switching times and the active mode at each time instant*;
- *estimate the state $x(t)$ of the system.*

3. Switching detection

A switching time t_s is a time instant at which system (1) changes from a mode, say, q to another $q^\star$. Due to the smoothness of the output function h, this change is not reflected in the output $y(t)$ but in its first derivative, since

$$y'(t_s^-) = L_{f_q} h(x(t_s)), \quad y'(t_s^+) = L_{f_{q^\star}} h(x(t_s)).$$

This fact makes the detection of the switching time t_s difficult.

In this work, the changes in the dynamics of (1) corresponding to the switching times are detected from the analysis of the second-order derivative of $y(t)$. This analysis is performed by means of the convolution in a moving time interval (sliding window) of $y(t)$ with suitable functions. Similar ideas are presented in Laboudi et al. (2019), Mboup (2007) and Fliess and Sira-Ramírez (2003) for switched linear systems. In these works, the linearity of the systems is instrumental.

3.1 Switching detection for a single output system

The switching detection strategy (estimator) for single output systems (1), i.e. when $h : \mathcal{X} \to \mathbb{R}$, is presented first, for the sake of clarity. The extension to multiple output systems, when $h : \mathcal{X} \to \mathbb{R}^p$, is shown in Section 3.3.

The only condition already imposed on the system, i.e. the joint observability, is rather general. In order to detect the occurrence of a switching, the following additional hypothesis about the behaviour of the output derivative is introduced.

Assumption 3.1 (Switching detectability): *There exists $\mu > 0$ such that for any pair $q, q^\star \in \mathcal{Q}$ with $q \neq q^\star$*

$$|L_{f_q} h(x) - L_{f_{q^\star}} h(x)| \geqslant \mu \quad \forall x \in \mathcal{X}. \tag{2}$$

Remark 3.1: Assumption 3.1 is rather strong and could be weakened by involving derivatives of $y(t)$ of higher order as, by example, replacing (2) with

$$|L^k_{f_q} h(x) - L^k_{f_{q^\star}} h(x)|$$
$$\geqslant \mu \text{ for each } x \in \mathcal{X}, \quad \text{with } k = k(x) \in \{2, \ldots, N-1\}.$$

Nevertheless, such weakening implies a considerable increase in the computational burden. So, from now on, we will suppose that Assumption 3.1 holds.

If (2) holds, then it is possible to detect a change of the mode of the system based on the first derivative of the output signal $y(t)$. In fact, (2) implies that at each switching time say, t_s, $y'(t)$ has a jump discontinuity with oscillation $|y'(t_s^+) - y'(t_s^-)|$ whose lower bound is μ. As a consequence, if (a) $y'(t)$ can be properly estimated and (b) jumps of $y'(t)$ greater or equal in absolute value to μ can be detected, then switching times can also be detected.

Remark 3.2: It is known that to synthesise and implement a differentiator to compute the derivative of a signal is not a trivial task. When the frequency spectrum characteristics of the signal to be processed are known, it is possible to build a system that approximates the transfer function of an ideal differentiator in the frequency range of the signal (Kumar & Roy, 1988; Pei & Shyu, 1988). In this case, low pass filters are used to reduce the effects of the noise present in the output.

The range of application of these methods is limited to a reduced number of output signals, and in addition, the internal dynamics of the system that approximates the differentiator imposes delays that degrade the estimate of t_s.

Differentiators based on sliding modes (Levant, 1998) increase the operating range and robustness against noise, but at the expense of allowing discontinuous actions and requiring more complex computational efforts to obtain good results in signal processing.

On the other hand, the numerical detection of the jump based on the first derivative of $y(t)$ is performed by comparing two values ($y'(t_s^+)$ and $y'(t_s^-)$), that in principle might be of several orders of magnitude greater than μ. In this case, the detection of t_s is a priori difficult.

Due to these facts, in this paper, we resort to the approximation of (generalised) derivatives of the second order of $y(t)$, with the aim to detect a spike at the switching time t_s, which is an easier task.

3.1.1 Generalised functions

Next, some results about generalised functions necessary for the development of the switching detection strategy are presented. For details, see, for example, Halperin and Schwartz (1952), Gel'fand and Shilov (1964) and Zemanian (2011).

Lemma 3.2: *Let $f(t)$ a continuous function whose first derivative $f'(t)$ is smooth, except at $t = t_s$ where it has a jump discontinuity. Then the generalised derivative of second order of $f(t)$ is given by*

$$D^{(2)} f(t) = F''(t) + [f'(t_s^+) - f'(t_s^-)]\delta(t - t_s), \tag{3}$$

where

$$F''(t) = \begin{cases} f''(t) & \text{if } t \neq t_s \\ 0 & \text{if } t = t_s \end{cases}$$

and $\delta(t)$ is the Dirac delta function.

Lemma 3.3: *Let $f(t)$, as in Lemma 3.2 and $t_a < t_b$ such that $t_s \neq t_a$ and $t_s \neq t_b$. Let also $I = [t_a, t_b]$ and $w(t) = \chi_I(t) f(t)$. Then the generalised derivative of second order of $w(t)$ is given by*

$$\begin{aligned} D^{(2)} w(t) = \chi_I(t) D^{(2)} f(t) + \sum_{k=0}^{1} f^{(k)}(t_a) \delta^{(1-k)}(t - t_a) \\ - f^{(k)}(t_b) \delta^{(1-k)}(t - t_b), \end{aligned} \tag{4}$$

where χ_I is the characteristic function of I, i.e.

$$\chi_I(t) = \begin{cases} 1 & \text{if } t \in I, \\ 0 & \text{if } t \notin I. \end{cases}$$

Lemma 3.4: *Let $\{\delta_m, m \in \mathbb{N}\}$ be a sequence of smooth functions such that $\lim_{m\to\infty} \delta_m = \delta$ in the generalised sense, and let w be a piecewise continuous function of compact support. Then the jth generalised derivative of $w(t)$ verifies*

$$D^{(j)} w(t) = \lim_{m\to\infty} \int_{-\infty}^{\infty} \delta_m^{(j)}(t - s) w(s)\, \mathrm{d}s. \tag{5}$$

3.1.2 Switching detection strategy

Following Laboudi et al. (2019), we compute $D^{(2)} y(t)$ by implementing a mobile window scheme, so that $y(t)$ is processed in time sections.

For a given time t_i and a fixed time interval τ (mobile time window width) let $t_{i+1} = t_i + \tau$, $I_i = [t_i, t_{i+1}]$ and $y_{\tau_i}(t) = \chi_{I_i}(t) y(t)$. In this case, according to (4),

$$\begin{aligned} D^{(2)} y_{\tau_i}(t) = \chi_{I_i}(t) D^{(2)} y(t) + \sum_{k=0}^{1} y^{(k)}(t_i) \delta^{(1-k)}(t - t_i) \\ - y^{(k)}(t_{i+1}) \delta^{(1-k)}(t - t_{i+1}). \end{aligned}$$

Suppose first that in I_i no switching occurs. Then

$$D^{(2)}y_{\tau_i}(t) = \chi_{I_i}(t)y''(t) + \sum_{k=0}^{1} y^{(k)}(t_i)\delta^{(1-k)}(t-t_i) - y^{(k)}(t_{i+1})\delta^{(1-k)}(t-t_{i+1}),$$

since in this case $D^{(2)}y(t) = y''(t)$.

Suppose now that a switching occurs at $t_s \in (t_i, t_{i+1})$. It follows that

$$\begin{aligned} D^{(2)}y_{\tau_i}(t) &= \chi_{I_i}(t)Y''(t) + [y'(t_s^+) - y'(t_s^-)]\delta(t-t_s) \\ &\quad + \sum_{k=0}^{1} y^{(k)}(t_i)\delta^{(1-k)}(t-t_i) \\ &\quad - y^{(k)}(t_{i+1})\delta^{(1-k)}(t-t_{i+1}), \end{aligned} \tag{6}$$

where, as above,

$$Y''(t) = \begin{cases} y''(t) & \text{if } t \neq t_s, \\ 0 & \text{if } t = t_s. \end{cases}$$

From (6) we establish a criterion to estimate the switching time t_s, which is based on searching for the term $[y'(t_s^+) - y'(t_s^-)]\delta(t-t_s)$ in successive time windows. If that term is not detected in, say, (t_i, t_{i+1}), the search continues in the next time window, which starts at time t_{i+1}, and so on.

3.2 Implementation

It must be pointed out that to obtain $D^{(2)}y_{\tau_i}(t)$ and to detect the switching times by means of (6), imply to construct a differentiator to compute the second derivative of a signal, and the difficulties mentioned in Remark 3.2 increase.

Hence, we present next a method based on Lemma 3.4 to approximate the values of $D^{(2)}y_{\tau_i}(t)$ $\forall t \in I_i$.

Consider the sequence of functions $\{\delta_m(t), m \in \mathbb{N}\}$ given by

$$\delta_m(t) = \frac{m}{\sqrt{2\pi}} \mathrm{e}^{-\frac{(mt)^2}{2}}. \tag{7}$$

It follows that $\lim_{m\to\infty} \delta_m = \delta$ in the generalised sense (Gel'fand & Shilov, 1964; Halperin & Schwartz, 1952; Zemanian, 2011).

Let for each $m \in \mathbb{N}$

$$w_m(t) = \int_{-\infty}^{\infty} y_{\tau_i}(\zeta)\delta_m''(t-\zeta)\,\mathrm{d}\zeta \tag{8}$$

$$= \int_{t_i}^{t_{i+1}} y(\zeta)\left(\frac{m^5}{\sqrt{2\pi}}\right)\mathrm{e}^{-\frac{m(t-\zeta)^2}{2}}\left((t-\zeta)^2 - \frac{1}{m^2}\right)\mathrm{d}\zeta. \tag{9}$$

Since $y_{\tau_i} * \delta_m''(t) = D^{(2)}y_{\tau_i} * \delta_m(t)$ holds (see Zemanian, 2011 for details), it follows from Lemma 3.4 that $\lim_{m\to\infty} w_m(t) = D^{(2)}y_{\tau_i}(t)$.

On the other hand, by (6)

$$\begin{aligned} w_m(t) &= D^{(2)}y_{\tau_i} * \delta_m(t) \\ &= [\chi_{I_i}(t)Y''] * \delta_m(t) + [y'(t_s^+) - y'(t_s^-)]\delta_m(t-t_s) \\ &\quad + \sum_{k=0}^{1} y^{(k)}(t_i)\delta_m^{(1-k)}(t-t_i) \\ &\quad - y^{(k)}(t_{i+1})\delta_m^{(1-k)}(t-t_{i+1}). \end{aligned} \tag{10}$$

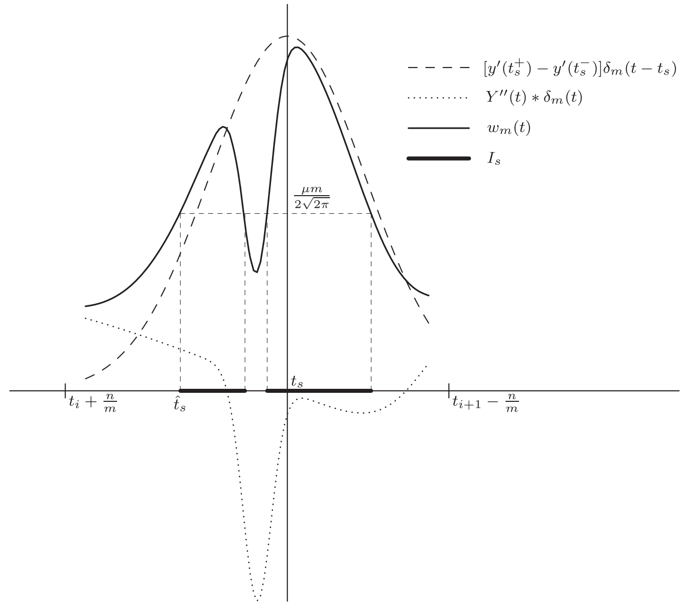


Figure 1. Switching detection scheme.

Since δ_m is a Gaussian function of zero mean and variance $\frac{1}{m}$, for any $t_0 \in \mathbb{R}$ 99.7 per cent of the useful information of $\delta_m(t-t_0)$ is concentrated in $|t-t_0| \leq \frac{3}{m}$.

Let $n \in \mathbb{N}$ and $\bar{I}_i = [t_i + \frac{n}{m}, t_{i+1} - \frac{n}{m}]$ with $n \geq 6$. Then from (10) and for all $t \in \bar{I}_i$ and m large enough, we have

$$\begin{aligned} w_m(t) &\cong Y'' * \delta_m(t) + [y'(t_s^+) - y_s'(t^-)]\delta_m(t-t_s) \\ &\cong Y''(t) + [y'(t_s^+) - y'(t_s^-)]\delta_m(t-t_s), \end{aligned} \tag{11}$$

since for those values of m, $Y'' * \delta_m(t) \cong Y'' * \delta(t) = Y''(t)$.

From (11) we can establish a numerical criterion to detect a switching in a time window as follows.

Since h and $f_q, q \in \mathcal{Q}$ are smooth and $\mathcal{X}$ is a compact set, there exists $M > 0$ such that $|Y''(t)| < M$ for all t. As $\max \delta_m = \frac{m}{\sqrt{2\pi}}$, if

$$M < \frac{\mu m}{2\sqrt{2\pi}}, \tag{12}$$

then $|w_m(t_s)| > \frac{\mu m}{2\sqrt{2\pi}}$. Fixed m that satisfies inequality (12) consider the open set $I_s = \{t \in \bar{I}_i \text{ suchthat } |w_m(t)| > \frac{\mu m}{2\sqrt{2\pi}}\}$ (see Figure 1). Then, the following switching detection condition (estimation of t_s) is established:

$$\hat{t}_s = \min\{t \in \partial I_s\}. \tag{13}$$

Remark 3.3: If $t_s \in [t_i, t_i + \frac{n}{m})$ or $t_s \in (t_{i+1} - \frac{n}{m}, t_{i+1}]$, $\hat{t}_s$ would not be computed.

In order to deal with this fact, a second switching detector is simultaneously used. This detector is similar to the first one, except that the intervals that define the moving windows are of the form $\overline{I^\star}_i = [t_i^\star + \frac{n}{m}, t_{i+1}^\star - \frac{n}{m}]$, with $t_i^\star = \frac{t_i + t_{i+1}}{2}$ and $t_{i+1}^* = t_i^\star + \tau$.

Remark 3.4: m must be selected as large as possible in order to fulfil the following requirements:

(1) δ_m approximates δ close enough so that $Y'' * \delta_m(t) \cong Y''(t)$,
(2) (12) holds,
(3) $t_i^\star + \frac{n}{m} < t_{i+1} - \frac{n}{m}$ and $t_{i+1} + \frac{n}{m} < t_{i+1}^\star - \frac{n}{m}$, and hence $m > \frac{4n}{\tau}$.

In principle, there is no theoretical restriction to the upper values that m and n can attain, but since the spectral bandwidth of δ_m is of the order of m^2, the greater is m the greater is the disruptive effect of the noise in the detection of the switching.

On the other hand, and from the point of view of numerical implementation, as m increases the computational cost also increases.

Remark 3.5: As for the selection of τ, in principle, it must hold that $\tau \ll \tau_D$ so as to detect just one switching at a time. Nevertheless, the switching can happen at any time of $\bar{I}_i$ and hence the condition (13) cannot be applied until (9) is evaluated for all $t \in \bar{I}_i$. This fact leads to a delay between the occurrence of the switching and its detection.

In consequence, τ must be selected small enough as to make this delay small, but no so small as to increase too much the computational burden.

3.3 Switching detection for multiple output systems

In the previous section, a scheme to detect a switching in a single output system was presented. In this section, we extend the strategy to multiple output systems ($y = h(x) = (h_1(x), \ldots, h_p(x)) \in \mathbb{R}^p$). To do so, we introduce the following hypothesis.

Assumption 3.5 (Switching detectability for multiple output systems): *There exists a family of compact sets $\{\mathcal{X}_k, k = 1, \ldots, p\}$ such that $\mathcal{X} = \cup_{k=1}^{p} \mathcal{X}_k$ and for each k a number $\mu_k > 0$ such that for any pair $q, q^\star \in \mathcal{Q}$ with $q \neq q^\star$*

$$|L_{f_q} h_k(x) - L_{f_{q^\star}} h_k(x)| \geqslant \mu_k \quad \forall x \in \mathcal{X}_k \tag{14}$$

for $k = 1, \ldots, p$.

Remark 3.6: This assumption is the extension of Assumption 3.1 to multiple output systems. Since the compact sets $\mathcal{X}_k$ are not necessarily disjoint, this assumption assures that at least one of the p outputs $y_k(t)$ verifies switching detectability condition (13) at any t_s.

The switching detection for multiple output systems is performed as follows: p switching detectors as above are set, one for each of the p outputs of the system. These detectors have the same values of τ and the same intervals I_i, but their respective values m_k differ, since they are computed according to (12) and the criteria established in Remark 3.4.

Suppose that a switching occurs at time $t_s \in \bar{I}_i$ and that $l \leq p$ of the detectors give each an estimate $\hat{t}_s^{k_j}$ according to (13), i.e.

$$\hat{t}_s^{k_j} = \min\{t \in \partial I_s^{k_j}\},$$

where

$$I_s^{k_j} = \left\{ t \in \bar{I}_i \text{ such that } |w_{m_{k_j}}^{k_j}(t)| > \frac{\mu_{k_j} m_{k_j}}{2\sqrt{2\pi}} \right\}$$

and

$$w_{m_{k_j}}^{k_j}(t) = \int_{t_i}^{t_{i+1}} y_{k_j}(\zeta) \left(\frac{m_{k_j}^5}{\sqrt{2\pi}} \right) e^{-\frac{m_{k_j}(t-\zeta)^2}{2}} \times \left((t-\zeta)^2 - \frac{1}{m_{k_j}^2} \right) d\zeta.$$

The estimate of the switching time t_s is obtained as

$$\hat{t}_s = \min\{\hat{t}_s^{k_j} : 1 \leq j \leq l\}. \tag{15}$$

4. State and mode estimation

The strategy for estimating states and modes of system (1) consists of the following steps:

(a) design an observer for each mode f_q and run all of them simultaneously (*bank of observers*);
(b) let a subset of subsystems (h, f_q) run at some specific times (*bank of subsystems*);
(c) estimate the current mode of the system;
(d) assign the state estimate given by the observer corresponding to the mode obtained in (c) as the estimate of the state of the system.

Similar strategies but using only a bank of observes were presented in Davila et al. (2012) and Barhoumi et al. (2012). In these works, the estimation of the mode depends on the type of observer implemented and of its dynamics.

4.1 Bank of observers

Several observers, based on different techniques, solve the problem of state estimation of nonlinear systems (Barbot et al., 2007; Bornard et al., 1995; Davila et al., 2009; Nicosia et al., 1994). Unlike the case of linear systems, the local observability condition (see, for instance, Hermann & Krener, 1977 for more details and precise definitions) does not guarantee the existence of an observer for a given nonlinear system. According to the technique selected for the design of the observer, the nonlinear system must verify additional conditions (see, e.g. Bornard et al., 1995; Gauthier et al., 1992).

In order to achieve some generalisation and based on the existing literature about observers' design for nonlinear systems, we consider a general structure for the observers of the bank, assuming that the requirements that this structure imposes on system (1) hold.

Assumption 4.1: *Given system* (1), *it is possible to design for each subsystem* (h, f_q), $q \in \mathcal{Q}$ *an observer whose dynamics is given by*

$$\left\{ \begin{array}{l} \dot{\hat{x}}_q = f_q(\hat{x}_q) - G_q(\hat{x}_q)K(\theta)[h(\hat{x}_q) - \mathcal{Y}(t, t_0, x_0, q)] \\ \hat{y}_q = h(\hat{x}_q) \end{array} \right. \tag{16}$$

that verifies for all $\theta > 0$

$$\begin{aligned} &\|\hat{x}_q(t) - \phi_q(t, t_0, x_0)\| \\ &\quad \leq \tilde{G}_q k(\theta)\, \mathrm{e}^{-\theta(t-t_0)} \|\hat{x}_q(t_0) - x_0\| \quad \forall t \geq t_0, \end{aligned} \tag{17}$$

where

- $G_q : \mathcal{X} \to \mathbb{R}^{n\times n}$ *is smooth and non-singular,*
- $K : \mathbb{R}_{>0} \to \mathbb{R}^{n\times p}$,
- $k(\theta)\, \mathrm{e}^{-\theta t} > 0$ *does not depend on* q *and for each* t *is strictly decreasing w.r.t.* θ,
- $\tilde{G}_q$ *is a bound related to* $G_q(\hat{x}_q)$.

Remark 4.1: In many of the observers found in the literature, G_q is related to the observability matrix $d\mathcal{O}_q$. As the terms of $d\mathcal{O}_q$ involve Lie derivatives of h with respect to f_q and since $\mathcal{X}$ is a compact set, it is possible to obtain $\tilde{G}_q$ as follows:

$$\overline{\lambda_q} = \max_{\hat{x}_q \in \mathcal{X}} \overline{\lambda}(G_q(\hat{x}_q)), \quad \underline{\lambda_q} = \min_{\hat{x}_q \in \mathcal{X}} \underline{\lambda}(G_q(\hat{x}_q)), \quad \tilde{G}_q = \frac{\overline{\lambda_q}}{\underline{\lambda_q}}. \tag{18}$$

We assume that (18) holds for the observer of Assumption 4.1.

Lemma 4.2: *Let for each* $q \in \mathcal{Q}$ *an observer whose dynamics is given by* (16). *Let also* $M > 0$ *and* $t_e > 0$. *Then there exists* $\theta = \theta^\star$ *(the same for all the observers) so that, for every* $q \in \mathcal{Q}$, *the following holds:*

$$\|h(\hat{x}_q(t)) - h(\phi_q(t, x_0))\| \leq M \quad \forall t \geq t_e. \tag{19}$$

Proof: Consider, with no loss of generality, $t_0 = 0$ and denote for each $q \in \mathcal{Q}$, $\phi_q(t, 0, \cdot) = \phi_q(t, \cdot)$.

Let for each $q \in \mathcal{Q}$, $\hat{x}_q(0) \in \mathcal{X}$ be the initial condition of the observer of mode q. Then, according to (17),

$$\|\hat{x}_q(t) - \phi_q(t, x_0)\| \leq \tilde{G}_q k(\theta)\, \mathrm{e}^{-\theta t} \|\hat{x}_q(0) - x_0\|; \quad \forall t \geq 0. \tag{20}$$

Consider the upper bound of the initial state error of the observer

$$\gamma_q = \max_{\bar{x} \in \partial\mathcal{X}} \{\|\hat{x}_q(0) - \bar{x}\|\}. \tag{21}$$

Also let

$$\tilde{G} = \max_{q \in \mathcal{Q}} \tilde{G}_q, \quad \alpha = \max_{\xi \in \mathcal{X}} \left\| \frac{\partial h}{\partial x}(\xi) \right\| \quad \text{and} \quad \gamma = \max_{q \in \mathcal{Q}} \{\gamma_q\}. \tag{22}$$

Then for all $t \geq 0$ and from (20), we have

$$\begin{aligned} &\|h(\hat{x}_q(t)) - h(\phi_q(t, x_0))\| \\ &\quad \leq \alpha \|\hat{x}_q(t) - \phi_q(t, x_0)\| \leq \alpha \tilde{G}_q k(\theta)\, \mathrm{e}^{-\theta t} \|\hat{x}_q(0) - x_0\| \\ &\quad \leq \alpha \tilde{G}_q \gamma_q k(\theta)\, \mathrm{e}^{-\theta t} \leq \alpha \tilde{G} \gamma k(\theta)\, \mathrm{e}^{-\theta t}. \end{aligned} \tag{23}$$

Since $k(\theta)\, \mathrm{e}^{-\theta t_e}$ is strictly decreasing with θ, there exists $\theta^\star$ such that

$$k(\theta^\star)\, \mathrm{e}^{-\theta^\star t_e} \leq \frac{M}{\alpha \tilde{G} \gamma}.$$

Let $\theta = \theta^\star$. From (23), we have for all $t \geq t_e$

$$\begin{aligned} \|h(\hat{x}_q(t)) - h(\phi_q(t, x_0))\| &\leq \alpha \tilde{G} \gamma k(\theta^\star)\, \mathrm{e}^{-\theta^\star t} \\ &= \alpha \tilde{G} \gamma k(\theta^\star)\, \mathrm{e}^{-\theta^\star t_e}\, \mathrm{e}^{-\theta^\star (t - t_e)} \\ &\leq \alpha \tilde{G} \gamma k(\theta^\star)\, \mathrm{e}^{-\theta^\star t_e} \leq M \end{aligned}$$

and the lemma holds. ■

Remark 4.2: Lemma 4.2 leads to a design condition that takes into account the hybrid characteristic of the system: it is imposed that $t_e \ll \tau_D/2$.

This requirement can be more or less conservative depending on the performance objectives that are required, but ensures that before a new switching occurs the observation error lies within a prescribed band. This fact is essential for the criterion developed for the mode selection.

4.2 Bank of subsystem and mode and state estimation

The bank of subsystems is formed by the models of (some of) the subsystems of (1), whose dynamics are given by

$$\left\{ \begin{array}{l} \dot{x}_q = f_q(x_q) \\ y_q = h(x_q) \end{array} \right. \quad \forall q \in \mathcal{Q}. \tag{24}$$

The algorithm implemented to estimate the system mode, $\hat{q}$, consists of the following steps:

(a) the comparison of the output errors between the output of each observer of the bank and the output of the system: $e_{y_q}(t) = \hat{y}_q(t) - y(t)$, and
(b) the comparison of the state errors between the state of each subsystem of the bank of subsystems and the state of the corresponding observer: $e_{x_q}(t) = \hat{x}_q(t) - x_q(t)$.

We denote with t_k, $k \in \mathbb{N}$ the time instants at which the observation strategy is implemented. The estimation of mode and states is obtained under two different circumstances: (a) in the initial interval with $t_1 = 0$ and (b) when $t_k = t_{i+1}$ for $\hat{t}_s \in I_i$ with $k > 1$, where $\hat{t}_s$ is the time instant when a switching is detected by either (13) or (15).

The algorithm for mode and state estimation proceeds as follows: let in each step $k \in \mathbb{N}$ of the algorithm $\Gamma_k \subset \mathcal{Q}$, M_k and t_k as defined below.

(I) At time t_k the bank of observers starts with the same initial condition $\hat{x}_q(t_k) = x^\star$ for all $q \in \Gamma_k$ and, according to (16), provides from each observer a state estimate $\hat{x}_q(t)$ and an output estimate $\hat{y}_q(t)$.

In the interval $[t_k, t_k + t_e]$, each mode generates a different output trajectory. At least one of the norms of the output errors $\|e_{y_q}(t_k + t_e)\|$ of the bank of observers (the one that corresponds to the active mode) will be under the upper bound M_k established in Lemma 4.2.

As $t_e \ll \tau_D$, in the interval $[t_k, t_k + t_e]$ the system evolves in the (unknown) mode $\sigma(t_k) = q_k$, so that the e_{y_q} are the representatives of each subsystem.

The set $\Gamma_k \subset \mathcal{Q}$ of the possible actives modes is redefined as

$$\Gamma_k = \{q \in \Gamma_k \text{ such that } \|e_{\hat{y}_q}(t_k + t_e)\| < M_k\} \tag{25}$$

(Ia) The redefined set $\Gamma_k = \{q^*\}$, then $\hat{q}_k = q^*$ and the estimate states of (1) is set as $\hat{x}(t) = \hat{x}_{\hat{q}}(t)$ for all $t \in [t_k + t_e, t_{k+1})$.

(Ib) The redefined set Γ_k has more than one element. Then a bank of subsystems as given by (24) with Γ_k instead of $\mathcal{Q}$ is set to evolve in the time period $[t_k + t_e, t_k + t_e + \Delta t]$ (with Δt given by Lemma A.1 in the appendix), with initial condition $x_q(t_k + t_e) = \hat{x}_q(t_k + t_e)$ for each subsystem $q \in \Gamma_k$.

If the conditions of Lemma A.1 in the appendix are satisfied, the mode estimation is carried out according to the rule

$$\hat{q}_k = \underset{q \in \Gamma_k}{\operatorname{argmin}} \|e_{x_q}(t_k + t_e + \Delta t)\| \tag{26}$$

and the estimate of the states of (1) is set as $\hat{x}(t) = \hat{x}_{\hat{q}}(t)$ for all $t \in [t_k + t_e + \Delta t, t_{k+1})$.

(II) When a new switching time t_s is detected by either (13) or (15) with $\hat{t}_s \in I_i$, the algorithm proceeds as follows: (i) k is increased to $k+1$, (ii) t_{k+1} is defined as $t_{k+1} = t_{i+1}$, (iii) the estimate of the states of (1) is set as $\hat{x}(t) = \hat{x}(t_{k+1})$ for $t \in [t_{k+1}, t_{k+1} + t_e)$ and eventually for $t \in [t_{k+1}, t_{k+1} + t_e + \Delta t)$ and (iv) since at t_{k+1} an estimate of the mode $\hat{q}_k$ already exists, Γ_{k+1} is defined as $\Gamma_{k+1} = \mathcal{Q} \backslash \{\hat{q}_k\}$.

Finally, (v) the bank of observers is redefined according to the new set Γ_{k+1}, (vi) the initialising state is taken as $x^\star = \hat{x}(t_{k+1})$ and (vii) M_{k+1} is determined from Lemma 4.2 with γ as given by (21) – (22) with $x^\star$ instead of $\hat{x}_q(0)$. Note that in this case $\gamma_q = \gamma$ for all $q \in \Gamma_{k+1}$.

Remark 4.3: The first step of the algorithm proceeds as follows: let the system and the algorithm start at $t = 0$. Take $t_1 = 0$, $\Gamma_1 = \mathcal{Q}$, $x^\star \in \mathcal{X}$ arbitrary and M_1 as determined from Lemma 4.2 where γ is given by (21) – (22) with $x^\star$ instead of $\hat{x}_q(0)$.

If the mode commutes at t_s and $\hat{t}_s < t_e$, or $\hat{t}_s < t_e + \Delta t$ in the case I(b), take $t_1 = \hat{t}_s$ and $x^\star$ and Γ_1 as above.

5. Example

In this section, we present an example in which the estimation scheme described in the previous sections was implemented.

Table 1. System constants.

$a_1 = 0.2$	$b_1 = 0.2$	$c_1 = 5.7$
$a_2 = 0.1$	$b_2 = 0.1$	$c_2 = 14$
$a_3 = 0.15$	$b_3 = 2$	$c_3 = 4$

Consider switched system (1) in $\mathbb{R}^3$ with three modes $\mathcal{Q} = \{1, 2, 3\}$ described by

$$\begin{cases} \dot{x}_1 = -x_2 - x_3, \\ \dot{x}_2 = x_1 + a_i x_2, \\ \dot{x}_3 = b_i + x_3(x_1 - c_i), \\ y = \tanh(x_2), \end{cases} \tag{27}$$

where a_i, b_i and c_i are constant parameters which take the values shown in Table 1.

The switching signal used in this example has a dwell-time $\tau_D = 1$ s and a length of 30 s. A detail of this signal is shown in Table 2.

On the other hand, as the output mapping $\mathcal{O}_q(x) = \{L^j_{f_q} h(x),\ j = 0, \ldots, 2\ q = 1, \ldots, 3, \}$ is

$$\mathcal{O}_q(x) = \begin{bmatrix} \tanh(x_2) \\ 0.1(x_1 + a_q x_2)\,\mathrm{sech}^2(x_2) \\ \mathrm{sech}^2(x_2)(-0.1(x_1 + x_3) + 0.1(x_1 + a_q x_2) \\ (0.1a_q - 0.2(x_1 + a_q x_2)\tanh(x_2))) \end{bmatrix},$$

$d\mathcal{O}_q(x)$ has full rank if $x_2 \neq 0$, and hence each subsystem is locally weakly observable. From the one-to-one relationship between y and x_2, it follows readily that the switched system is generically jointly observable.

Given that the system verifies

$$\begin{aligned} &|L_{f_i} h(x) - L_{f_j} h(x)| \\ &\quad = 0.1(1 - \tanh^2(x_2))|x_2|\,|a_i - a_j| \quad \forall i, j \in \mathcal{Q}, \end{aligned} \tag{28}$$

and since $1 - \tanh^2(x_2) \neq 0\ \forall x_2 \in \mathbb{R}$ then, according to (2), it is possible to detect a switching only when the state $x_2 \neq 0$. It follows that (2) will only hold if the values of x are restricted to a set such that $|x_2| \geq \mu_\star$ for some $\mu_\star$.

As this condition is not a priori feasible, in this example we resorted to an alternative computation of μ in (2), based on the one-to-one relationship between x_2 and y. In fact, from (28) we obtain

$$\begin{aligned} &|L_{f_i} h(x) - L_{f_j} h(x)| \\ &\quad = 0.1(1 - y^2)|\mathrm{arctanh}(y)|\,|a_i - a_j| \quad \forall i, j \in \mathcal{Q}. \end{aligned}$$

From this condition, an adaptive bound is obtained as follows: for each step i of the switching detection algorithm let

$$\mu_i = 0.1|1 - \bar{y}^2|\,|\mathrm{arctanh}(\bar{y})|\mu^\star \text{ with } \bar{y} = \max_{t \in \bar{I}_i}\{y(t)\}, \tag{29}$$

where $\mu^\star$ is a design parameter. We note that the bounds μ_i so obtained are associated with the window scheme of the switching detection.

In simulations, this strategy improved, in terms of computational burden, the switching detection task. The value adopted for $\mu^\star$ was 0.05.

Table 2. Switching signal.

t_s [seg]	0	2.1245	6.1024	11.7298	14.9385	17.6751	20.5002	24.8296	26.9388
$\sigma(t)$	2	3	1	2	3	1	3	2	3

Other parameters not related to the bank of observers were chosen as $\tau = 1 \times 10^{-2}$ s, $m = 1 \times 10^4$ and $n = 8$. These parameters fulfil some of the requirements of Remarks 3.4 and 3.5.

As for the bank of observers, a high gain observer presented in Gauthier et al. (1992) was implemented for each subsystem. This kind of observers verifies Assumption 4.1.

In the design of the observer $\theta = 10$ was selected, with the objective to fulfil the condition of Remark 4.2 for $t_e = 0.1$ s.

Remark 5.1: Prior to the implementation of the state estimation scheme, the unknown parameters $\mathcal{X}$ and Δt were estimated by simulation.

(1) $\mathcal{X}$ was estimated as follows. It was assumed that all the possible initial states of the system belong to a sphere $\mathcal{X}_0$ of radius 3 centred at $x_0 = (3.9,\ -3.2,\ 0.03)$ and initial conditions $\{x_0^j, j = 1, \ldots, 300\}$ were chosen at random in $\mathcal{X}_0$. Also switching signals $\{\sigma_i, i = 1, \ldots, 50\}$ with dwell time $\tau_D = 1$ s and length 30 s each were randomly generated. For each initial condition and each switching signal, a trajectory $x(t, \sigma_i, x_0^j)$ was generated and the value $x_i^j = \max_{0 \le t \le 30} \|x(t, \sigma_i, x_0^j)\|$ computed. Finally, from

$$\bar{x} = \frac{1}{15000} \sum_{j=1}^{50} \sum_{i=1}^{300} x_i^j = 9.2518 \quad \text{and}$$

$$\sigma_x = \sqrt{\frac{1}{15000} \sum_{j=1}^{50} \sum_{i=1}^{300} (x_i^j - \bar{x})^2} = 5.0981,$$

an estimate

$$\mathcal{X} = \{x \in \mathbb{R}^3 : \|x\| \le \bar{x} + 3\sigma_x = 24.65\} \tag{30}$$

was obtained.

(2) The estimate of Δt was obtained as follows; 2000 points $\{x_0^k, k = 1, \ldots, 2000\}$ were selected at random in $\mathcal{X}$, and in a sphere of radius 0.01 centred in each x_0^k three points $\{x_j^k, j = 1, \ldots, 3\}$ were randomly selected. The points x_0^k played the role of x_0 in Figure A1, and the points x_j^k (where the sub-index j refers to the jth mode of the system) the role of $x_{q_0^\star}$ or x_{q_0}. Once a mode $q^\star \in \mathcal{Q}$ was selected, and for each k, the trajectory of system (1) starting in this mode at x_0^k was generated. The trajectories of the subsystems and of their corresponding observers in the banks, starting at $x_j^k, j = 1, \ldots, 3$, were also generated. The generation of the trajectories continued until inequality (A2) was verified for the first time, at $\delta t_k^{q^\star}$ s. This process was repeated three times, each one corresponding to a different selection of $q^\star$. Finally, from

$$\bar{\delta t} = \frac{1}{6000} \sum_{q^\star \in \mathcal{Q}} \sum_{k=1}^{2000} \delta t_k^{q^\star} = 0.0382$$

and

$$\sigma_{\delta t} = \sqrt{\frac{1}{6000} \sum_{q^\star \in \mathcal{Q}} \sum_{k=1}^{2000} (\delta t_k^{q^\star} - \bar{\delta t})^2} = 0.052,$$

we obtained the estimate

$$\Delta t = \bar{\delta t} + 3\sigma_{\delta t} = 0.1912 \text{ s}. \tag{31}$$

The parameters for the simulation of the state estimation scheme were taken as follows. Time of simulation: 30 s, initial state and initial estimated state: $x_0 = (3.9,\ -3.2,\ 0.03)$ and $\hat{x}_0 = (4.1,\ -3,\ 0.5)$, respectively.

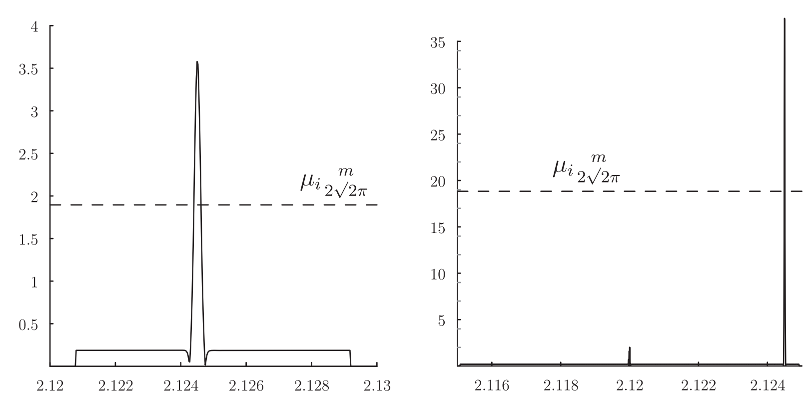
- In simulations, it was found that the values of M_k computed from (21) to (22) with $\mathcal{X}$ given by (30) are too conservative. It was also found that the following scheme not only simplified the task of evaluating M_k, but also improved the performance of the algorithm.

  M_{k+1} was defined as follows: let $\tilde{t}$ be the time at which the current mode of the system, $\hat{q}$, is estimated. Then,

  (1) if $\tilde{t} = t_k + t_e$, let $M_{k+1} = \tilde{M}\, \mathrm{e}^{-\tilde{\theta}(t_{k+1} - t_k)}$, where $\tilde{M} = \min_{q \in \Gamma_k - \{\hat{q}\}} \|\hat{x}_{\hat{q}}(\tilde{t}) - \hat{x}_q(\tilde{t})\|$,

  (2) if $\tilde{t} = t_k + t_e + \Delta t$, let $M_{k+1} = \tilde{M}\, \mathrm{e}^{-\tilde{\theta}(t_{k+1} - t_k - \Delta t)}$, where $\tilde{M} = \|e_{x_{\hat{q}}}(\tilde{t})\|$,

  with $\tilde{\theta} < \theta$, so that the dynamics of the variation of M_k be slower than the dynamics of the estimation of the observers.



(a) (b)

Figure 2. Detection of a switching for different values of m. (a) $m = 1 \times 10^4$, (b) $m = 1 \times 10^5$.

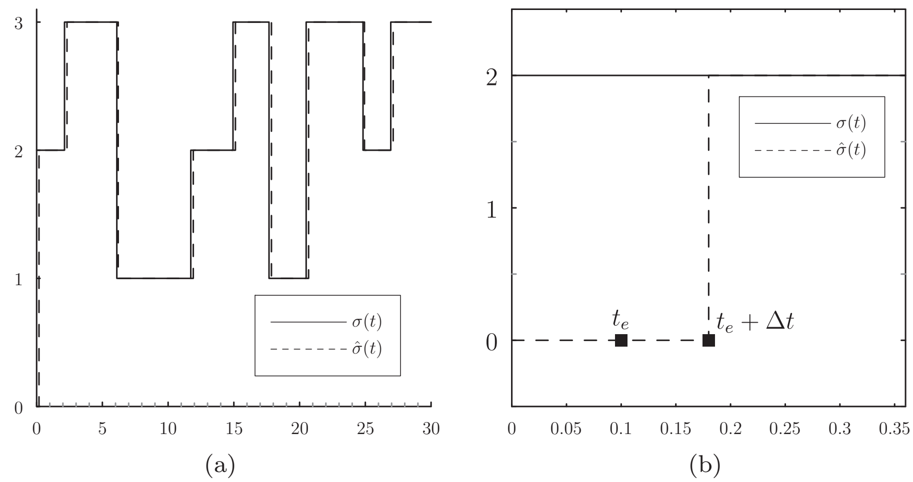


Figure 3. Reconstruction of the switching signal and detail at $t = 0$. (a) Switching signal reconstruction and (b) detail at $t = 0$.

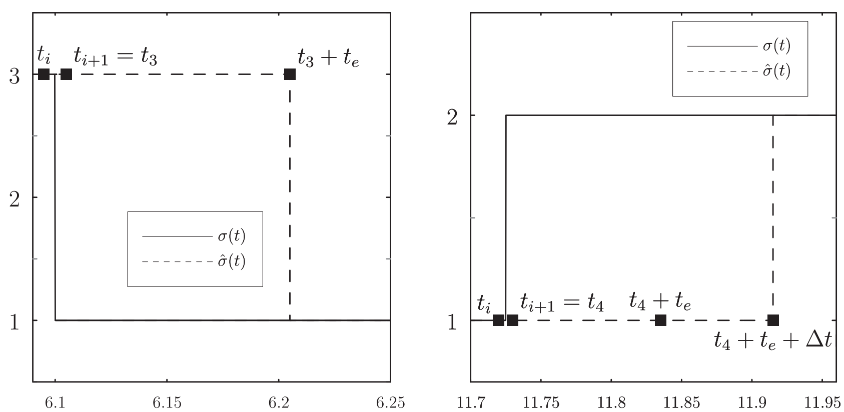


Figure 4. Mode detection for different values of $\hat{t}_s$. (a) Only the bank of observers evolves. (b) Both the bank of observers and the bank of subsystems evolve.

Once t_{k+1} is determined by the algorithm, the value of M_{k+1} is established.

The parameter $\tilde{\theta} = 1$ was taken and $M_1 = 1$ was computed as stated in Remark 4.3.

- Also in simulations, it was found that Δt, as given by (31), is too conservative, being the (finally adopted) value $\Delta t = 0.08$ s a suitable one.

Remark 5.2: Although theoretically sound, criterion (13) is not easy to apply, since the set I_s is difficult to compute. So in the simulations, the following switching detection condition was used instead:

$$\hat{t}_s = \min\left\{ t \in \bar{I}_i : |w_m(t)| \geq \frac{\mu m}{2\sqrt{2\pi}} \right\}. \tag{32}$$

Table 3. Simulation results.

$\hat{t}_s$	t_i	$t_{i+1} = t_k$	μ_i	M_k	$\tilde{M}$	$\tilde{t}$
2.1245	2.12	2.13	4.7506e−4	2.7324	4.3243e−4	2.31
6.1024	6.095	6.1050	7.2853e−4	8.7970e−6	0.0789	6.205
11.7298	11.725	11.735	3.1579e−4	2.8302e−4	0.0014	11.9150
14.9385	14.93	14.94	8.6780e−4	6.0285e−5	0.0022	15.12
17.6751	17.67	17.68	9.3783e−4	1.5483e−4	0.0015	17.86
20.5002	20.495	20.505	9.7347e−5	9.5257e−5	6.7435e−4	20.685
24.8296	24.825	24.835	5.1524e−4	9.6190e−6	0.1928	24.935
26.9388	26.93	26.94	4.9905e−4	0.0235	8.2714e−4	27.12

The results of the simulation of the state estimation scheme are shown in the figures below.

Figure 2 presents the detection of the change from mode $q = 2$ to mode $q = 3$, at $t_s = 2.1245$ s (see Table 2). The criterion of detection used was (32) for $m = 10^4$ and $m = 10^5$. We note that as m increases, an improvement of the estimation $\hat{t}_s$ of t_s is obtained.

Figure 3(a) shows $\sigma(t)$ and its estimate $\hat{\sigma}(t)$, and in Figure 3(b), a detail of their behaviour at $t = 0$ is presented. For $t \in$

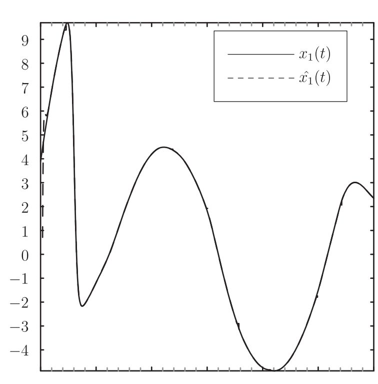


Figure 5. Time evolution of x_1 and $\hat{x}_1$.

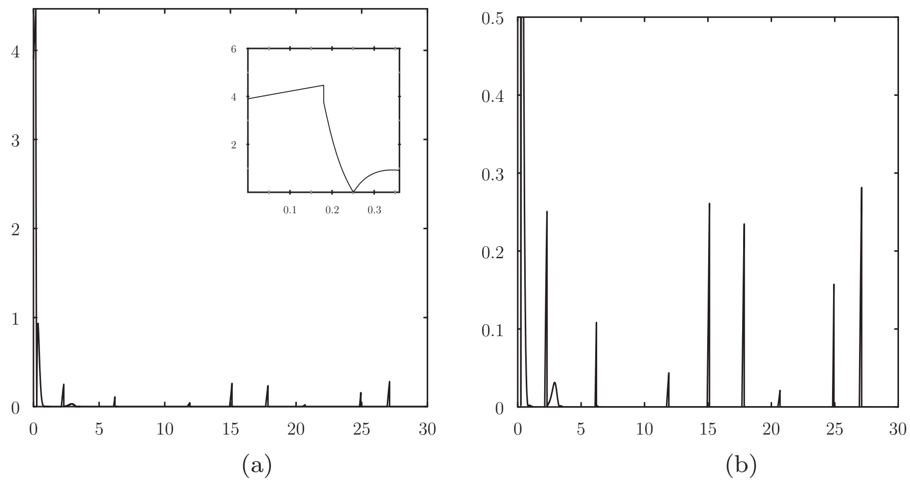


Figure 6. Norm of the estimation error for the state x_1. (a) Time evolution of $|\hat{x}_1 - x_1|$ and (b) detail.

$[0\,;\, t_e + \Delta t)$, $\hat{\sigma} = 0$ since no information of the switching signal was available. Note that, given the values of M_1 and $\tilde{\theta}$, once time t_e has elapsed, Γ_1 has more than one element. Hence the bank of subsystems must evolve in order to estimate the current mode. Also note the delay with which the algorithm estimates the current mode.

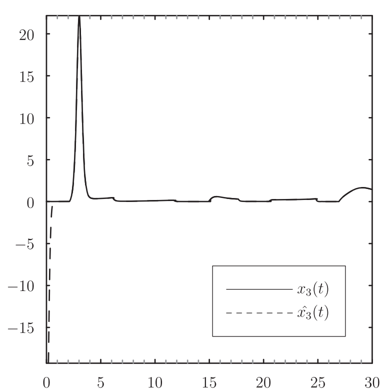


Figure 7. Time evolution of x_3 and $\hat{x}_3$.

This behaviour can also be seen in greater detail in Figure 4, where the two possible cases of mode estimation are presented. In Figure 4(a), the change of mode from $q = 3$ to mode $q = 1$ at $t_s = 6.1024$ s and the estimation of the new mode are shown. After I_τ where t_s takes place and is detected, Γ_3 has just one element and the new mode is estimated at $t_3 + t_e$. In Figure 4(b), the change of mode from $q = 1$ to $q = 2$ at $t_s = 11.7298$ s and the estimation of $\hat{q}_4$ are exhibited. In this case, Γ_4 has more than one element and then the new mode is estimated at time $t_4 + t_e + \Delta t$.

In Table 3, we present additional information that shows how the algorithm of detection of the switching and the values of some of the parameters evolve. We note that the switching time estimations $\hat{t}_s$ coincide up to the fourth decimal with the true switching times t_s as given by Table 2. This fact reflects the effectiveness of the proposed scheme.

The complete estimation process can be seen in Figures 5–8. In Figure 5, the evolution of $x_1(t)$ and $\hat{x}_1(t)$ is presented, while in Figure 6 the estimation error for this state is shown. In this figure the degradation of the estimation error behaviour near the switching times can be seen. This is due to the fact that the

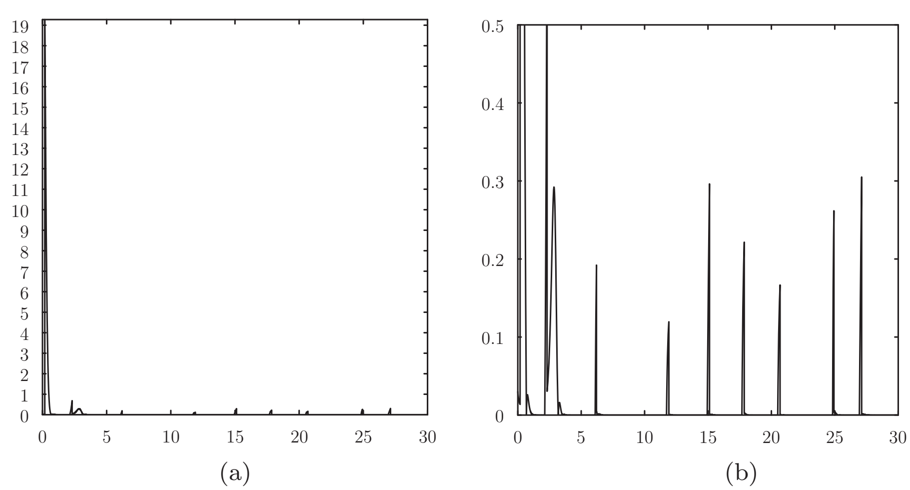


Figure 8. Norm of the estimation error for the state x_3. (a) Time evolution $|\hat{x}_3 - x_3|$ and (b) detail.

estimation of the states remains fixed during the periods $t_k + t_e$ and/or $t_k + t_e + \Delta t$. It can also be noted that once the current mode of the system is determined, the exponential decay of the estimation error allows achieving a good estimate of the state.

The estimation of state x_3 is presented in Figures 7 and 8, where a behaviour similar to that of the estimation of state x_1 can be seen.

Referring to the state x_2 no figure is shown, since x_2 is in a one-to-one relationship with the output.

6. Conclusion

In this work, an observation strategy for autonomous nonlinear switched systems was presented. The strategy is based on the measurement of the output signal and assumes that no information about the dynamics of the switching signal is available.

To develop the strategy, (a) a moving time window scheme was designed to detect the switching times via the convolution of the output and suitable functions and (b) a criterion based on the use of a bank of observers and a bank of subsystems was established that enables the estimation of the current mode of the switched system. The only condition that this criterion imposes on the model of the state observer is that the output error of the observer must obey a controlled exponential decay law.

Finally, an example was presented where the behaviour of the strategy is shown. In this example, one of the state observers already found in the literature and that meets the requirements of the mode detector was used.

Disclosure statement

No potential conflict of interest was reported by the author(s).

References

Alessandri, A., & Coletta, P. (2001).*Switching observers for continuous-time and discrete-time linear systems*. Proceedings of the American Control Conference, Arlington, VA (Vol. 3, pp. 2516–2521).

Barbot, J. P., Fliess, M., & Floquet, T. (2007). *An algebraic framework for the design of nonlinear observers with unknown inputs*. Proceedings of the IEEE Conference on Decision and Control, New Orleans, LA (pp. 384–389).

Barhoumi, N., Msahli, F., Djemaï, M., & Busawon, K. (2012). Observer design for some classes of uniformly observable nonlinear hybrid systems. *Nonlinear Analysis: Hybrid Systems*, *6*(4), 917–929. https://doi.org/10.1016/j.nahs.2012.03.002

Bornard, G., Celle-Couenne, F., & Gilles, G. (1995). Observability and observers. In A. J. Fossard & D. Normand-Cyrot (Eds.), *Nonlinear systems* (pp. 173–216). Springer. https://doi.org/10.1007/978-1-4615-2047-4_6

Ciccarella, G., Dalla Mora, M., & Germani, A. (1993). A Luenberger-like observer for nonlinear systems. *International Journal of Control*, *57*(3), 537–556. https://doi.org/10.1080/00207179308934406

Davila, J., Fridman, L., Pisano, A., & Usai, E. (2009). Finite-time state observation for non-linear uncertain systems via higher-order sliding modes. *International Journal of Control*, *82*(8), 1564–1574. https://doi.org/10.1080/00207170802590531

Davila, J., Ríos, H., & Fridman, L. (2012). State observation for nonlinear switched systems using nonhomogeneous high-order sliding mode observers. *Asian Journal of Control*, *14*(4), 911–923. https://doi.org/10.1002/asjc.516

Decarlo, R. A., Branicky, M. S., Pettersson, S., & Lennartson, B. (2000). Perspectives and results on the stability and stabilizability of hybrid systems. *Proceedings of the IEEE*, *88*(7), 1069–1082. https://doi.org/10.1109/5.871309

Fliess, M., & Sira-Ramírez, H. (2003, January). An algebraic framework for linear identification. *ESAIM: Control, Optimisation and Calculus of Variations*, *9*, 151–168. http://www.esaim-cocv.org/10.1051/cocv:2003008

Gauthier, J. P., Hammouri, H., & Othman, S. (1992). A simple observer for nonlinear systems applications to bioreactors. *IEEE Transactions on Automatic Control*, *37*(6), 875–880. https://doi.org/10.1109/9.256352

Gel'fand, I. M., & Shilov, G. E. (1964). *Generalized functions* (No. v. 1). AMS Chelsea Publishing, American Mathematical Society.

Gomez-Gutierrez, D., Ramirez-Trevino, A., Ruiz-Leon, J., & Di Gennaro, S. (2012). On the observability of continuous-time switched linear systems under partially unknown inputs. *IEEE Transactions on Automatic Control*, *57*(3), 732–738. https://doi.org/10.1109/TAC.2012.2224257

Halperin, I., & Schwartz, L. (1952). *Introduction to the theory of distributions*. University of Toronto Press.

Hermann, R., & Krener, A. J. (1977). Nonlinear controllability and observability. *IEEE Transactions on Automatic Control*, *22*(5), 728–740. https://doi.org/10.1109/TAC.1977.1101601

Karami, A., Yazdanpanah, M. J., & Moshiri, B. (2019). Robust switching signal estimation for a class of uncertain nonlinear switched systems. *International Journal of Control*, *92*(5), 1094–1102. https://doi.org/10.1080/00207179.2017.1382719

Kumar, B., & Dutta Roy, S. C. (1988). Design of digital differentiators for low frequencies. *Proceedings of the IEEE*, *76*(3), 287–289. https://doi.org/10.1109/5.4408

Laboudi, K., Messai, N., Manamanni, N., & Mboup, M. (2019). Fault and switching instants estimation for switched linear systems. In O. Boubaker, Q. Zhu, M. S. Mahmoud, J. Ragot, H. R. Karimi, & J. Dávila (Eds.), *New trends in observer-based control* (pp. 407–438). Elsevier.

Lee, C., Ping, Z., & Shim, H. (2013, July). *On-line switching signal estimation of switched linear systems with measurement noise*. European Control Conference (ECC) (pp. 2180–2185). IEEE.

Levant, A. (1998). Robust exact differentiation via sliding mode technique. *Automatica*, *34*(3), 379–384. https://doi.org/10.1016/S0005-1098(97)00209-4

Liberzon, D. (2003). *Switching in systems and control*. Birkhäuser.

Liberzon, D., & Morse, A. S. (1999). Basic problems in stability and design of switched systems. *IEEE Control Systems*, *19*(5), 59–70. https://doi.org/10.1109/37.793443

Lin, H., & Antsaklis, P. J. (2009). Stability and stabilizability of switched linear systems: A survey of recent results. *IEEE Transactions on Automatic Control*, *54*(2), 308–322. https://doi.org/10.1109/TAC.2008.2012009

Mboup, M. (2007). *Parameter estimation via differential algebra and operational calculus* (Research Report). https://hal.inria.fr/inria-00138294

Ngoc Dinh, T., & Defoort, M. (2019). Fixed-time state estimation for a class of switched nonlinear time-varying systems. *Asian Journal of Control*, *22*(1), 1782–1790. https://doi.org/10.1002/asjc.2068

Nicosia, S., Tornambè, A., & Valigi, P. (1994, September). Nonlinear map inversion via state observers. *Circuits, Systems and Signal Processing*, *13*, 571–589. https://doi.org/10.1007/BF02523184

Pei, S. C., & Shyu, J. J. (1988). Design of FIR Hilbert transformers and differentiators by eigenfilter. *IEEE Transactions on Circuits and Systems*, *35*(11), 1457–1461. https://doi.org/10.1109/31.14474

Petterson, S. (2006). *Designing switching observers for switched systems using multiple Lyapunov functions and dwell-time switching. IFAC Proceedings Volumes, 39*(5), 18–23. https://doi.org/10.3182/20060607-3-IT-3902.00008

Rios, H., Davila, J., & Fridman, L. (2012, January). *High-order sliding modes observers for linear autonomous-switched systems with unknown inputs*. 12th International Workshop on Variable Structure Systems, Mumbai, Maharashtra, India (pp. 428–433).

Shorten, R., Wirth, F., Mason, O., Wulff, K., & King, C. (2007). Stability criteria for switched and hybrid systems. *SIAM Review*, *49*(4), 545–592. https://doi.org/10.1137/05063516X

Tanwani, A., Shim, H., & Liberzon, D. (2013). Observability for switched linear systems: Characterization and observer design. *IEEE Transactions on Automatic Control*, *58*(4), 891–904. https://doi.org/10.1109/TAC.2011.2166669

Tian, Y., Floquet, T., Belkoura, L., & Perruquetti, W. (2009). Algebraic switching time identification for a class of linear hybrid systems. *IFAC Proceedings Volumes*, *42*(17), 404–408. https://doi.org/10.3182/20090916-3-ES-3003.00069

Vidal, R., Chiuso, A., Soatto, S., & Sastry, S. (2003). Observability of linear hybrid systems. In O. Maler & A. Pnueli (Eds.), *Hybrid systems: Computation and control* (Lecture Notes in Computer Science, Vol. 2623, pp. 526–539. Springer. https://doi.org/10.1007/3-540-36580-X_38

Xie, G., & Wang, L. (2004). Necessary and sufficient conditions for controllability and observability of switched impulsive control systems. *IEEE Transactions on Automatic Control*, *49*(6), 960–966. https://doi.org/10.1109/TAC.2004.829656

Zemanian, A. H. (2011). *Distribution theory and transform analysis: An introduction to generalized functions, with applications*. Dover Publications.

Appendix. Lemma for state and mode estimation

Lemma A.1: *Let system* (1) *in the mode* $q^\star \in \mathcal{Q}$ *with the initial condition* $x(t_0) = x_0$. *Let in addition* $e_{x_q}(t) = \hat{x}_q(t) - x_q(t)$ *be the estimation error at time t between each subsystem* (16) *and the corresponding observer* (24) *for the same initial condition* $x_q(t_0) = \hat{x}_q(t_0)$. *If* $\forall q \in \Gamma - \{q^\star\}$, *it holds that*

$$\|K(\theta)[h(x_{q^\star}(t_0)) - h(x_0)]\| < \frac{\lambda_{\min}}{\lambda_{\max}} \|K(\theta)[h(x_q(t_0)) - h(x_0)]\| \quad \text{(A1)}$$

with $\lambda_{\min} = \min_{q\in\Gamma} \underline{\lambda_q}$, $\lambda_{\max} = \max_{q\in\Gamma} \overline{\lambda_q}$ *and* $\underline{\lambda_q}, \overline{\lambda_q}$ *as in* (18), *then there exists* $\Delta t > 0$ *not depending on* t_0 *such that* $\forall t \in (t_0, t_0 + \Delta t]$

$$\|e_{x_{q^\star}}(t)\| < \|e_{x_q}(t)\|. \quad \text{(A2)}$$

Proof: The dynamics of the error of each subsystem, e_q, verifies

$$\begin{aligned}
\dot{e}_{x_q}(t) &= f_q(\hat{x}_q(t)) - f_q(x_q(t)) - G_q(\hat{x}_q(t))K(\theta)[h(\hat{x}_q(t)) - \mathcal{Y}(t, t_0, x_0, q)] \\
&= f_q(\hat{x}_q(t)) - f_q(x_q(t)) - G_q(\hat{x}_q(t))K(\theta)[h(\hat{x}_q(t)) \\
&\quad - h(x_q(t)) + h(x_q(t)) - \mathcal{Y}(t, t_0, x_0, q)] \\
&= f_q(\hat{x}_q(t)) - f_q(x_q(t)) - G_q(\hat{x}_q(t))K(\theta)[h(\hat{x}_q(t)) - h(x_q(t))] \\
&\quad - G_q(\hat{x}_q(t))K(\theta)[h(x_q(t)) - \mathcal{Y}(t, t_0, x_0, q)] \\
&= -G_q(\hat{x}_q(t))K(\theta)[h(x_q(t)) - \mathcal{Y}(t, t_0, x_0, q)], \qquad \text{(A3)}
\end{aligned}$$

since x_q the dynamics of the subsystem q and that of the corresponding observer $\hat{x}_q$ are the same as both of them evolve from the same initial condition x_{q_0}.

Consider the first-order approximation of $e_{x_q}(t)$ around t_0

$$\begin{aligned}
e_{x_q}(t) &= e_{x_q}(t_0) + \dot{e}_{x_q}(t_0)(t - t_0) + \Upsilon(t - t_0) \\
&= -G_q(\hat{x}_q(t_0))K(\theta)[h(x_q(t_0)) - h(x_0)](t - t_0) + \Upsilon(t - t_0) \quad \text{(A4)}
\end{aligned}$$

as $e_{x_q}(t_0) = 0$. Here $\Upsilon(t - t_0)$ are the terms of order higher than one.

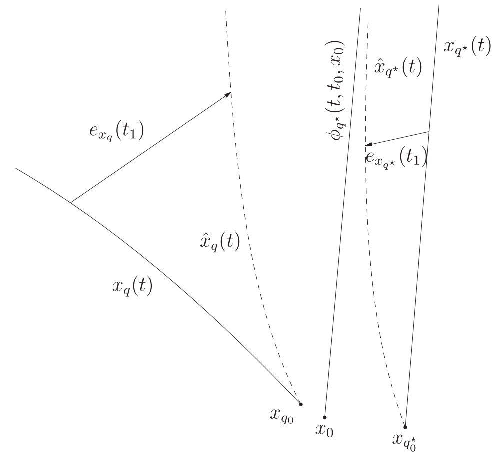


Figure A1. Phase portrait of system (1), bank of observers (16) and bank of subsystem (24).

Due to the smoothness of f_q and h, and to the compactness of $\mathcal{X}$, $\|\Upsilon(t - t_0)/(t - t_0)^2\|$ is uniformly bounded in closed intervals, say $[t_0, t^\star]$ (with bound depending on $t^\star$). It follows that there exists $\Delta t > 0$ such that for any $t_1 \in (t_0, t_0 + \Delta t]$

$$e_{x_q}(t_1) \cong -G_q(\hat{x}_q(t_0))K(\theta)[h(x_q(t_0)) - h(x_0)](t_1 - t_0). \quad \text{(A5)}$$

Hence

$$\begin{aligned}
\|e_{x_{q^\star}}(t_1)\|^2 &\cong \|G_{q^\star}(\hat{x}_{q^\star}(t_0))K(\theta)[h(x_{q^\star}(t_0)) - h(x_0)]\|^2 \Delta t_1^2 \\
&\le \lambda_{\max}^2 \|K(\theta)[h(x_{q^\star}(t_0)) - h(x_0)]\|^2 \Delta t_1^2 \\
&< \lambda_{\min}^2 \|K(\theta)[h(x_q(t_0)) - h(x_0)]\|^2 \Delta t_1^2 \\
&\le \|G_q(\hat{x}_q(t_0))K(\theta)[h(x_q(t_0)) - h(x_0)]\|^2 \Delta t_1^2 \\
&\cong \|e_{x_q}(t_1)\|^2. \qquad \text{(A6)}
\end{aligned}$$

In Figure A1, a diagram of the relative positions of the trajectories involved is shown.