On a goodness-of-fit test for normality with unknown parameters and type-II censored data

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We propose a new goodness-of-fit test for normal and lognormal distributions with unknown parameters and type-II censored data. This test is a generalization of Michael's test for censored samples, which is based on the empirical distribution and a variance stabilizing transformation. We estimate the parameters of the model by using maximum likelihood and Gupta's methods. The quantiles of the distribution of the test statistic under the null hypothesis are obtained through Monte Carlo simulations. The power of the proposed test is estimated and compared to that of the Kolmogorov–Smirnov test also using simulations. The new test is more powerful than the Kolmogorov–Smirnov test in most of the studied cases. Acceptance regions for the PP, QQ and Michael's stabilized probability plots are derived, making it possible to visualize which data contribute to the decision of rejecting the null hypothesis. Finally, an illustrative example is presented.

Keywords: Kolmogorov–Smirnov test; maximum likelihood and Gupta's estimators; Monte Carlo simulation; PP, QQ and stabilized probability plots

1. Introduction

A model of reference in statistics is the normal distribution, which has dominated the landscape of distribution theory and statistical applications for over 100 years. Today, the remarkable properties of this distribution are well-known and widely used. Many statistical models and their optimal properties rely in some way on the assumption of normality. When this hypothesis cannot be sustained, several alternatives may be undertaken. Among them we can mention: (i) to transform the data to obtain normality or (ii) to model directly the data by an appropriate distribution for the random variable (r.v.) of interest. A well-known model that is used as an alternative to the normal distribution upon non-negative support and positive skewness in several applications, for example in lifetime analysis, is the lognormal (LN) one, which has cumulative distribution function (CDF) given by

$$F_X(x) = \Phi\left(\frac{\log(x) - \mu}{\sigma}\right); \quad x > 0,\ \mu \in \mathbb{R},\ \sigma > 0, \tag{1}$$

where $\exp(\mu)$ and σ are its scale and shape parameters, respectively, and $\Phi(\cdot)$ is the standard normal CDF. The notation $X \sim \mathrm{LN}(\mu, \sigma^2)$ is used in this case. Thus, the r.v. $Y = \log(X)$ follows the normal distribution with mean μ and variance σ^2, which is denoted by $Y \sim \mathrm{N}(\mu, \sigma^2)$. For more details, see [16, p. 78].

An important topic of statistical application is the analysis of censored data. These kinds of data are usually found in lifetime studies when the experiment ends before all the units present an event of interest [2,6–8,14,22]. In particular, when a parametric model is utilized in this context, it is of interest to have inferential and visual goodness-of-fit procedures to validate this model.

The Kolmogorov–Smirnov (KS) test is a well-known goodness-of-fit method, the statistic of which is denoted by D. The KS test is based on the comparison between the empirical cumulative distribution function (ECDF) and a theoretical CDF specified in the null hypothesis (H_0). A graph that allows one to visualize the coherence of the ECDF with a specified theoretical CDF is the PP plot, so that it can be associated with the KS test. Analogously, empirical quantiles (ordered observations) can be compared to theoretical quantiles producing the QQ plot. A disadvantage of the PP and QQ plots is that some of their points are more variable than others. Michael [17] modified the KS statistics, D, using the arcsin transformation for stabilizing the variance of the plotted points. The probability plot related to this variance stabilizing transformation is known as the stabilized probability (SP) plot. A test associated with the SP plot was proposed by Michael [17], the statistic of which is denoted by D_{SP}.

In goodness-of-fit tests for a completely specified distribution, one can test any continuous distribution as long as its parameters are known and without loss of generality to suppose in H_0 the uniform distribution on [0, 1], which is denoted by U(0, 1). Michael [17] studied the power of the test based on D_{SP} for a completely specified distribution and proved that this is more powerful than the KS test for certain distributions in the alternative hypothesis (H_1). In the case of censored samples, modifications of KS and Michael's tests for a completely specified distribution can be revised in [5,7,11].

A more realistic situation is presented when the parameters of the distribution specified in H_0 are unknown so that they must be estimated. In this case, the distribution of the statistic of the goodness-of-fit test depends on the parameter estimators, the estimation method, and the sample size, as well as on the distribution specified in H_0. However, when the distribution specified in H_0 is in the location–scale family and these location and scale parameters are estimated by appropriate methods, the distribution of the goodness-of-fit statistics does not depend on the true values of the unknown parameters; see [10, p. 102]. Particularly, Lilliefors [9,15] modified the KS test for testing normality with unknown parameters. Michael [17] proposed this same modification for D_{SP}. The KS test for normality with unknown parameters can also be modified for censored data estimating the parameters by means of, for example, the maximum likelihood (ML) method. Nevertheless, in this case, the ML procedure does not provide analytical expressions for the parameter estimators so that iterative numerical techniques must be used. For this reason, the ML method was discarded in the past, so that tests for normality with unknown parameters and censored data were based on linear estimators, as those proposed by Gupta [12], which are easily computed and have been shown to be asymptotically efficient [1]. Recently, Sultan and Khaleel [26] proposed tests for normality and censored data with parameters estimated by Gupta's method. They estimated the CDF by the kernel nonparametric method instead of using the ECDF as an estimator of the CDF.

In this article, we introduce a new test based on D_{SP} for normality with unknown parameters and right type-II censored data. The parameters are estimated using ML and Gupta's methods. In addition, probability plots and their acceptance regions are provided. These regions make possible to visualize which data contribute to the decision of rejecting H_0. The methodology presented here for right type-II censoring is also valid for censoring to the left and for the LN distribution; for more details, see Remark 2.2. In right type-II censoring, the uncensored observations keep the same position that they would have if all the observations were uncensored. This aspect allows us to construct goodness-of-fit tests for censored data in an analogous way to that with uncensored data.

In Section 2, the new goodness-of-fit test and a modification of the KS test for normality with unknown parameters and censored data are introduced. The computation algorithm, tables of critical points, formulas for PP, QQ and SP plots and acceptance regions for these plots are also presented in this section. In Section 3, a comparison between the powers of the proposed test and the KS test is presented. In Section 4, for the purposes of illustration, an example of the obtained results is considered. Finally, some conclusions are drawn.

2. The new goodness-of-fit test for censored data

Let $\boldsymbol{X} = [X_1, \ldots, X_n]^\top$ be a random sample of size n from a distribution with CDF $F(\cdot)$. As is well-known, if $U_j = F(X_j)$, for $j = 1, \ldots, n$, then $\boldsymbol{U} = [U_1, \ldots, U_n]^\top$ is a random sample of size n from the U(0, 1) distribution. If $F(\cdot)$ belongs to the location–scale family with parameters μ (location) and σ (scale), $F(x) = G([x - \mu]/\sigma)$, and $G(\cdot)$ denotes the central CDF, then we have

$$U_j = G\left(\frac{X_j - \mu}{\sigma}\right) \quad \text{and} \quad U_{(j)} = G\left(\frac{X_{(j)} - \mu}{\sigma}\right), \quad j = 1, \ldots, n, \tag{2}$$

where $X_{(1)}, \ldots, X_{(n)}$ and $U_{(1)}, \ldots, U_{(n)}$ denote the order statistics from the samples $\boldsymbol{X}$ and $\boldsymbol{U}$, respectively. Michael's statistic is defined as

$$D_{\mathrm{SP}} = \max_{1 \le j \le n} \left\{ \frac{2}{\pi} \left| \arcsin\left(\sqrt{\frac{j - 0.5}{n}}\right) - \arcsin\left(\sqrt{U_{(j)}}\right) \right| \right\}. \tag{3}$$

The reason postulated by him for defining the SP plot and D_{SP} was based on the fact that, for $U \sim \mathrm{U}(0, 1)$, the r.v. $S = [2/\pi] \arcsin(\sqrt{U})$ follows the sine distribution on (0, 1), which is denoted by SIN(0, 1), whose density is $f_S(s) = [\pi/2] \sin(\pi s)$, for $0 < s < 1$. The order statistics of a random sample of size n from the SIN(0, 1) distribution, denoted by $S_{(1)}, \ldots, S_{(n)}$, have a constant asymptotic variance, due to that, as n approaches to ∞ and j/n approaches to q, $\mathrm{Var}[n\, S_{(j)}]$ approaches to $1/\pi^2$, which is independent of q, for $j = 1, \ldots, n$. Michael's SP graph is obtained by plotting the points $[[2/\pi] \arcsin(\sqrt{[j - 0.5]/n}), [2/\pi] \arcsin(\sqrt{u_{(j)}})]$, for $j = 1, \ldots, n$.

Consider a random sample and the hypotheses H_0: "the sample is drawn from a normal distribution with parameters μ and σ" against H_1:"the sample is not drawn from this normal distribution", i.e.,

$$\mathrm{H} : F(x) = G\left(\frac{x - \mu}{\sigma}\right) \equiv \Phi\left(\frac{x - \mu}{\sigma}\right) \quad \text{versus} \quad \mathrm{H}_1 : F(x) \neq \Phi\left(\frac{x - \mu}{\sigma}\right). \tag{4}$$

If the distribution in H_0 given in Equation (4) is completely specified, then the expression for D_{SP} given in Equation (3) can be used with $U_{(j)} = \Phi([X_{(j)} - \mu]/\sigma)$, for $j = 1, \ldots, n$.

Remark 2.1 For testing the hypotheses given in Equation (4) with unknown location (μ) and scale (σ) parameters, μ and σ must be replaced by their respective estimators $\hat{\mu}$ and $\hat{\sigma}$. However, in this case, even when H_0 is true, the corresponding $\hat{U}_{(j)} = F_0([X_{(j)} - \hat{\mu}]/\hat{\sigma})$, for $j = 1, \ldots, n$, is not an ordered uniform sample and so the distribution of D_{SP} with $U_{(j)}$ replaced by $\hat{U}_{(j)}$ differs from the distribution of this statistic when the parameters are known. The quantiles of the distribution of D_{SP} based on $\hat{U}_{(j)}$ under H_0 were obtained by Michael [17] through simulation.

2.1 *Modified statistics for censored samples*

To contrast the hypotheses given in Equation (4) in the case of unknown parameters and right type-II censored data, let $U_{(1)} < \cdots < U_{(r)} = T$ be the uncensored observations of the censored (whole) random sample of size n $(r \leq n)$. In this case, T is an r.v., r is fixed, $(n - r)$ observations are greater than T, and the proportion of uncensored observations is $p = r/n$. We propose the following modification for D_{SP}:

$$D^{\star}_{\mathrm{SP}} = \max_{1 \leq j \leq r} \left\{ \frac{2}{\pi} \left| \arcsin\left(\sqrt{\frac{j - 0.5}{n}}\right) - \arcsin\left(\sqrt{\hat{U}_{(j)}}\right) \right| \right\}, \tag{5}$$

where $\hat{U}_{(j)} = \Phi([X_{(j)} - \hat{\mu}]/\hat{\sigma})$, for $j = 1, \ldots, r$. In this case, the ML estimates of μ and σ are obtained from

$$\hat{\mu} = \bar{X} + \lambda(p, \hat{\xi})[T - \bar{X}] \quad \text{and} \quad \hat{\sigma}^2 = S^2 + \lambda(p, \hat{\xi})[T - \bar{X}]^2, \tag{6}$$

respectively, where

$$\bar{X} = \sum_{j=1}^{r} \frac{X_j}{r}, \quad S^2 = \sum_{j=1}^{r} \frac{[X_j - \bar{X}]^2}{r}, \quad \lambda(p, \xi) = \frac{Y(p, \xi)}{Y(p, \xi) + \xi}, \quad Y(p, \xi) = \frac{\phi(\xi)[p - 1]}{\Phi(-\xi)p},$$

and $\xi = [T - \mu]/\sigma$, with $\phi(\cdot)$ being the standard normal density. As mentioned earlier, the ML estimates obtained from Equation (6) must be computed using iterative numerical methods. If $p = 1$, then $Y(p, \xi) = 0$ and so $\lambda(p, \xi) = 0$, obtaining thus the ML estimators of μ and σ of the normal distribution for uncensored random samples. If $p < 1$, we must first solve a nonlinear equation in ξ by using iterative numerical methods to obtain $\hat{\xi}$, then evaluate $\lambda(p, \hat{\xi})$, and finally compute $\hat{\mu}$ and $\hat{\sigma}^2$; for more details about this procedure, see [6,8]. For the KS test, we propose to modify D using ML estimates for the unknown parameters with right type-II censored data, i.e.,

$$D^{\star} = \max_{1 \leq j \leq r} \left\{ \frac{2}{\pi} \left| \frac{j - 0.5}{n} - \hat{U}_{(j)} \right| \right\} + \frac{0.5}{n}, \tag{7}$$

where $\hat{U}_{(j)}$ is defined as in $D^{\star}_{\mathrm{SP}}$ given in Equation (5). We call $D^{\star}_{\mathrm{SP}}$ and $D^{\star}$ modified because they must be evaluated at the estimates of μ and σ.

2.2 *Computation algorithm*

For testing the hypotheses given in (4) based on the statistic defined in Equation (5), the following steps must be done:

(1) Compute the ML estimates of μ and σ, say $\hat{\mu}$ and $\hat{\sigma}$, using Equation (6).
(2) Obtain $\hat{Z}_{(j)} = [X_{(j)} - \hat{\mu}]/\hat{\sigma}$, for $j = 1, \ldots, r$.
(3) Determine $\hat{U}_{(j)} = \Phi(\hat{Z}_{(j)})$, for $j = 1, \ldots, r$.
(4) Calculate $D^{\star}_{\mathrm{SP}}$, which we denote by $d^{\star}_{\mathrm{SP}}$, by using the observed value of $\hat{U}_{(j)}$ obtained in (3).

Table 1. Quantiles of the distribution of $D^{\star}_{\mathrm{SP}}$ for a normal distribution under H_0 with parameters estimated by ML method and right type-II censoring for the indicated values of p, n, and $1-\alpha$.

p	n	$d^{\star}_{\mathrm{SP}}(0.50)$	$d^{\star}_{\mathrm{SP}}(0.75)$	$d^{\star}_{\mathrm{SP}}(0.90)$	$d^{\star}_{\mathrm{SP}}(0.95)$	$d^{\star}_{\mathrm{SP}}(0.99)$
0.3	20	0.0497	0.0629	0.0748	0.0825	0.0977
	25	0.0469	0.0583	0.0698	0.0772	0.0913
	30	0.0462	0.0570	0.0677	0.0744	0.0900
	40	0.0426	0.0523	0.0621	0.6888	0.0825
	50	0.0404	0.0490	0.0578	0.0638	0.0761
	60	0.0382	0.0462	0.0543	0.0601	0.0710
	70	0.0362	0.0438	0.0517	0.0568	0.0685
	80	0.0350	0.0421	0.0498	0.0546	0.0648
	90	0.0337	0.0407	0.0480	0.0528	0.0625
	100	0.0328	0.0395	0.0462	0.0506	0.0597
0.6	20	0.0637	0.0770	0.0904	0.0992	0.1178
	25	0.0598	0.0719	0.0840	0.0923	0.1107
	30	0.0564	0.0680	0.0796	0.0869	0.1020
	40	0.0516	0.0617	0.0721	0.0792	0.0943
	50	0.0479	0.0573	0.0670	0.0738	0.0873
	60	0.0450	0.0537	0.0628	0.0683	0.0812
	70	0.0424	0.0508	0.0597	0.0653	0.0769
	80	0.0406	0.0484	0.0571	0.0625	0.0742
	90	0.0393	0.0469	0.0547	0.0601	0.0711
	100	0.0377	0.0449	0.0522	0.0575	0.0691
0.8	20	0.0705	0.0845	0.0983	0.1080	0.1263
	25	0.0656	0.0785	0.0909	0.0994	0.1187
	30	0.0618	0.0737	0.0855	0.0935	0.1096
	40	0.0561	0.6671	0.0778	0.0849	0.1001
	50	0.0520	0.0617	0.0717	0.0789	0.0916
	60	0.0486	0.0574	0.0669	0.0732	0.0867
	70	0.0457	0.5448	0.0633	0.0690	0.0810
	80	0.4376	0.0517	0.0602	0.0660	0.7883
	90	0.0422	0.0500	0.0579	0.0633	0.0742
	100	0.0404	0.0477	0.0556	0.0606	0.0724

(5) Compare $d^{\star}_{\mathrm{SP}}$ with the suitable quantile given in Table 1.
(6) Reject H_0 at the α level of significance if $d^{\star}_{\mathrm{SP}}$ is greater than the $(1-\alpha)$th quantile of the distribution of $D^{\star}_{\mathrm{SP}}$, which we denote by $d^{\star}_{\mathrm{SP}}(1-\alpha)$.

Remark 2.2 Note the following:

(1) An analogous algorithm to that described in steps (1)–(6) must be applied for testing the hypotheses given in Equation (4) based on $D^{\star}$, which is defined in Equation (7). In this case, Table 2 must be used for obtaining the suitable quantiles.
(2) Gupta's estimates can also be used in Equations (5) and (7). In this case, the distributions of $D^{\star}_{\mathrm{SP}}$ and $D^{\star}$ are different from those obtained by ML estimation so that the corresponding quantiles must be estimated. In the case of Equation (5), quantiles for some values of n and p can be found in [6] and those corresponding to the statistic given in Equation (7) in [10].
(3) D'Agostino and Stephens [10] suggested using the same approximate quantiles of quadratic-type goodness-of-fit statistics with right type-II censored data for right type-I censoring. Tests obtained in such a way have an approximate level. They suggested doing so in the case of a large sample size and $p > 0.2$. The same suggestion could be used for the proposed tests because to the best of our knowledge, there are not exact tests available in this case.

Table 2. Quantiles of the distribution of $D^\star$ for a normal distribution under H_0 with parameters estimated by ML method and right type-II censoring for the indicated values of p, n, and $1-\alpha$.

p	n	$d^\star(0.50)$	$d^\star(0.75)$	$d^\star(0.90)$	$d^\star(0.95)$	$d^\star(0.99)$
0.3	20	0.0801	0.0938	0.1094	0.1186	0.1363
	25	0.0693	0.0819	0.0961	0.1047	0.1200
	30	0.0655	0.0784	0.0913	0.1001	0.1161
	40	0.0571	0.0679	0.0799	0.0876	0.1026
	50	0.0514	0.0612	0.0716	0.0783	0.0932
	60	0.0470	0.0562	0.0658	0.0719	0.0845
	70	0.0432	0.0516	0.0606	0.0667	0.0793
	80	0.0408	0.0487	0.0572	0.0628	0.0732
	90	0.0385	0.0463	0.0545	0.0594	0.0698
	100	0.0368	0.0439	0.0516	0.0564	0.0657
0.6	20	0.1104	0.1298	0.1505	0.1643	0.1890
	25	0.0988	0.1168	0.1354	0.1472	0.1719
	30	0.0901	0.1074	0.1247	0.1361	0.1591
	40	0.0789	0.0932	0.1083	0.1177	0.1394
	50	0.0707	0.0837	0.0973	0.1069	0.1264
	60	0.0648	0.0766	0.0892	0.0973	0.1152
	70	0.0600	0.0710	0.0828	0.0909	0.1060
	80	0.0561	0.0666	0.0777	0.0852	0.0996
	90	0.0534	0.0634	0.0740	0.0809	0.0951
	100	0.0506	0.0600	0.0698	0.0762	0.0893
0.8	20	0.1233	0.1452	0.1684	0.1827	0.2120
	25	0.1105	0.1298	0.1505	0.1633	0.1936
	30	0.1013	0.1191	0.1381	0.1496	0.1743
	40	0.0880	0.1041	0.1201	0.1311	0.1531
	50	0.0790	0.0929	0.1083	0.1178	0.1395
	60	0.0723	0.0853	0.0985	0.1076	0.1260
	70	0.0670	0.0791	0.0918	0.1001	0.1159
	80	0.0625	0.0742	0.0861	0.0942	0.1106
	90	0.0599	0.0707	0.0822	0.0893	0.1059
	100	0.0566	0.0668	0.0772	0.0845	0.0997
	100	0.0539	0.0635	0.0740	0.0810	0.0956

Table 3. Formulas for constructing the indicated plots with censored data.

Plot	Ordinate	Abscissa
PP	$u_j = \Phi\left(\dfrac{x_{(j)} - \hat{\mu}}{\hat{\sigma}}\right)$	$v_j = \dfrac{j - 0.5}{n}$
QQ	$x_{(j)}$	$y_j = \Phi^{-1}\left(\dfrac{j - 0.5}{n}\right)$
SP	$s_j = \dfrac{2}{\pi} \arcsin\left(\sqrt{\Phi\left(\dfrac{x_{(j)} - \hat{\mu}}{\hat{\sigma}}\right)}\right)$	$w_j = \dfrac{2}{\pi} \arcsin\left(\sqrt{\dfrac{j - 0.5}{n}}\right)$

(4) Note that if $X \sim \mathrm{N}(\mu, \sigma^2)$, then $-X \sim \mathrm{N}(-\mu, \sigma^2)$. Thus, for a left censored random sample, changing the sign of every observation, a right censored random sample is obtained. Then, the tests proposed in this article can also be used for left type II censored samples from a normal distribution.

(5) By using the relationship between the normal and LN models given in Equation (1), the proposed tests can be adapted for testing lognormality.

Table 4. $100[1-\alpha]\%$ acceptance regions using quantiles $d^\star(1-\alpha)$ and $d^\star_{\mathrm{SP}}(1-\alpha)$.

Plot	Statistic	Lines defining acceptance regions
PP	$D^\star$	$\left[\max\left\{v - d^\star(1-\alpha) + \frac{0.5}{n}, 0\right\}, \quad \min\left\{v + d^\star(1-\alpha) - \frac{0.5}{n}, 1\right\}\right]$
PP	$D^\star_{\mathrm{SP}}$	$\left[\max\left\{\left[\sin\left(\arcsin\left(\sqrt{v}\right) - \frac{\pi}{2} d^\star_{\mathrm{SP}}(1-\alpha)\right)\right]^2, 0\right\}, \quad \min\left\{\left[\sin\left(\arcsin\left(\sqrt{v}\right) + \frac{\pi}{2} d^\star_{\mathrm{SP}}(1-\alpha)\right)\right]^2, 1\right\}\right]$
QQ	$D^\star$	$\left[\max\left\{\hat{\mu} + \hat{\sigma}\,\Phi^{-1}\left(\Phi(y) - d^\star(1-\alpha) + \frac{0.5}{n}\right), 0\right\}, \quad \min\left\{\hat{\mu} + \hat{\sigma}\,\Phi^{-1}\left(\Phi(y) + d^\star(1-\alpha) - \frac{0.5}{n}\right), 1\right\}\right]$
QQ	$D^\star_{\mathrm{SP}}$	$\left[\max\left\{\hat{\mu} + \hat{\sigma}\,\Phi^{-1}\left(\left[\sin\left(\arcsin\left(\sqrt{\Phi(y)}\right) - \frac{\pi}{2} d^\star_{\mathrm{SP}}(1-\alpha)\right)\right]^2\right), 0\right\}, \quad \min\left\{\hat{\mu} + \hat{\sigma}\,\Phi^{-1}\left(\left[\sin\left(\arcsin\left(\sqrt{\Phi(y)}\right) + \frac{\pi}{2} d^\star_{\mathrm{SP}}(1-\alpha)\right)\right]^2\right), 1\right\}\right]$
SP	$D^\star$	$\left[\max\left\{\frac{2}{\pi}\arcsin\left(\sqrt{\left[\sin\left(\frac{\pi}{2}w\right)\right]^2 - d^\star(1-\alpha) + \frac{0.5}{n}}\right), 0\right\}, \quad \min\left\{\frac{2}{\pi}\arcsin\left(\sqrt{\left[\sin\left(\frac{\pi}{2}w\right)\right]^2 + d^\star(1-\alpha) - \frac{0.5}{n}}\right), 1\right\}\right]$
SP	$D^\star_{\mathrm{SP}}$	$\left[\max\left\{w - d^\star_{\mathrm{SP}}(1-\alpha), 0\right\}, \quad \min\left\{w + d^\star_{\mathrm{SP}}(1-\alpha), 1\right\}\right]$

2.3 $D^{\star}_{\mathrm{SP}}$ *quantiles*

We have obtained quantiles of the distribution of $D^{\star}_{\mathrm{SP}}$ under the null hypothesis given in Equation (4) by simulation. For several sample sizes (n), proportions of uncensored observations (p), and levels of significance (α), 10,000 independent samples have been generated. The quantiles have been obtained for values of $n = 20, 25$ and from 30 to 100 by 10, p from 0.2 to 1 by 0.1, and $\alpha = 0.01, 0.05, 0.10, 0.25, 0.50$. (Of course $p = 1$ leads to the quantiles of the distribution of $D^{\star}_{\mathrm{SP}}$ in the case of uncensored samples.) For reasons of space, only quantiles for some selected values of n, p, and α are given in Table 1. More complete tables can be requested from the authors; see also [6].

2.4 *PP, QQ and SP plots and acceptance regions using* $D^{\star}_{\mathrm{SP}}$

To obtain acceptance regions on PP, QQ and SP plots, the quantiles of the distributions of $D^{\star}$ and $D^{\star}_{\mathrm{SP}}$ must be used. Formulas for constructing these plots are shown in Table 3, while Table 4 summarizes expressions for determining the corresponding $100[1-\alpha]\%$ acceptance regions. In all the formulas presented in these tables, r is the number of uncensored observations, n is the whole sample size, $j = 1, \ldots, r$, and $\Phi^{-1}(\cdot)$ is the inverse standard normal CDF. If the r uncensored observations lie within the constructed regions, then H_0 cannot be rejected at the α level of significance.

Remark 2.3 As mentioned, acceptance regions on PP, QQ and SP plots for single (left or right) censoring can be analogously obtained as those of the uncensored case. The censored observations do not appear in the proposed plots so that only the uncensored portion of the observations from the hypothetical distribution is plotted.

3. Power study

To evaluate and compare the powers of the proposed tests, we have conducted an extensive Monte Carlo simulation study. As in Section 2.3, 10,000 independent samples were generated for several sample sizes, proportions of uncensored observations, and two levels of significance. We have considered two estimation methods (ML and Gupta) and diverse distributions as alternative hypotheses. The results are summarized in Tables 5–7 and in Figures 1–3. (More complete results can be obtained upon request or from [6].) This study allows us to draw the following conclusions:

(1) When the ML estimation method is used, as expected, for both proposed tests and for every distribution considered in H_1, the power increases as the sample size increases. When the proportion of uncensored observations increases, the power increases too for every distribution except for the U(0, 1) one. In this case, the power function is not a monotone function of p and this fact will require further studies. Under H_0, the empirical power, as expected, is close to the nominal level. This can be seen in the last row of the panel corresponding to each value of p in Tables 5–7.

(2) When the ML estimation method is used, the test based on $D^{\star}_{\mathrm{SP}}$ is more powerful than that based on $D^{\star}$ for almost all the considered sample sizes, proportions of uncensored observations, and alternative hypotheses. An exception occurs with the double exponential (DE) distribution, since here the test based on $D^{\star}_{\mathrm{SP}}$ is more powerful only for some values of p and n, although the differences are small. For the exponential (EXP), U(0, 1) and $\chi^2(4)$ distributions, the power of the test based on $D^{\star}_{\mathrm{SP}}$ is much greater than that of the KS test, especially for large

Table 5. Estimated power (in %) of the tests based on the indicated statistics, distributions and values of p and n using $\alpha = 0.05$ with parameters estimated by ML method.

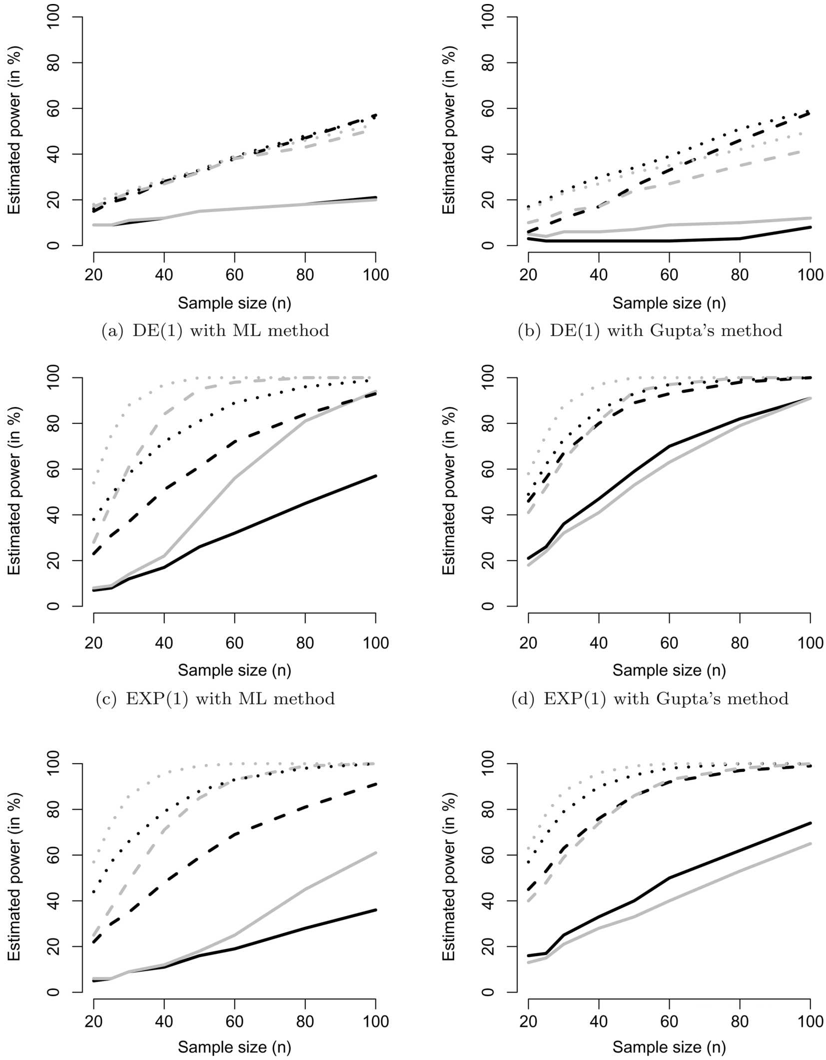
		$n = 20$		$n = 25$		$n = 30$		$n = 40$		$n = 50$		$n = 60$		$n = 80$		$n = 100$	
p	Distribution	$D^{\star}$	$D^{\star}_{SP}$	$D^{\star}$	$D^{\star}_{SP}$	$D^{\star}$	$D^{\star}_{SP}$	$D^{\star}$	$D^{\star}_{SP}$	$D^{\star}$	$D^{\star}_{SP}$	$D^{\star}$	$D^{\star}_{SP}$	$D^{\star}$	$D^{\star}_{SP}$	$D^{\star}$	$D^{\star}_{SP}$
0.3	DE(1)	9	9	9	9	10	11	12	12	15	15	16	16	18	18	21	20
	EXP(1)	7	8	8	9	12	14	17	22	26	39	32	56	45	81	57	94
	LN(0, 1)	5	6	6	6	9	9	11	12	16	18	19	25	28	45	36	61
	$t(1)$	39	38	45	44	55	55	68	68	78	77	84	84	93	92	96	95
	$t(3)$	13	13	13	14	16	18	20	22	25	26	29	31	35	37	41	42
	U(0, 1)	6	7	7	7	10	11	12	16	18	27	23	40	31	67	41	85
	$\chi^2(1)$	17	21	21	25	32	40	47	69	63	89	72	96	87	100	95	100
	$\chi^2(3)$	6	6	6	6	7	8	11	11	14	17	18	26	25	47	34	66
	$\chi^2(4)$	5	5	5	5	6	6	8	7	10	11	12	16	17	28	23	42
	N(0, 1)	5	5	5	5	5	5	5	5	5	5	5	4	5	4	5	5
0.6	DE(1)	15	17	19	20	21	23	28	27	32	32	38	38	47	43	57	51
	EXP(1)	23	28	31	45	37	61	51	84	61	95	72	98	84	100	93	100
	LN(0, 1)	22	25	30	37	35	49	48	71	59	85	69	93	81	99	91	100
	$t(1)$	60	62	69	71	76	78	87	88	92	93	96	96	99	99	100	100
	$t(3)$	15	19	19	22	22	26	28	31	32	36	38	42	44	48	53	55
	U(0, 1)	10	13	13	19	15	27	21	46	25	64	32	81	42	95	53	99
	$\chi^2(1)$	52	70	65	88	74	96	89	100	95	100	98	100	100	100	100	100
	$\chi^2(3)$	15	17	19	24	22	34	32	54	39	72	49	85	61	97	74	99
	$\chi^2(4)$	12	12	15	17	17	23	24	36	28	49	35	65	47	83	57	94
	N(0, 1)	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
0.8	DE(1)	16	18	20	22	23	24	28	29	33	33	39	39	48	46	56	53
	EXP(1)	38	54	49	75	58	88	72	97	81	100	89	100	96	100	99	100
	LN(0, 1)	44	57	57	74	66	86	79	96	88	99	93	100	98	100	100	100
	$t(1)$	63	66	72	74	78	80	88	89	93	93	96	96	99	99	100	100
	$t(3)$	16	19	20	23	23	26	27	31	33	36	38	42	44	48	52	57
	U(0, 1)	9	13	13	21	14	28	18	47	21	65	27	81	35	95	44	99
	$\chi^2(1)$	70	92	83	98	90	100	97	100	99	100	100	100	100	100	100	100
	$\chi^2(3)$	25	32	32	49	39	62	50	83	61	93	71	98	83	100	91	100
	$\chi^2(4)$	19	23	25	34	30	45	38	63	47	78	55	89	68	98	79	100
	N(0, 1)	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5

Table 6. Estimated power (in %) of the tests based on the indicated statistics, distributions and values of p and n using $\alpha = 0.01$ with parameters estimated by ML method.

		$n=20$		$n=25$		$n=30$		$n=40$		$n=50$		$n=60$		$n=80$		$n=100$	
p	Distribution	$D^\star$	$D^\star_{\mathrm{SP}}$	$D^\star$	$D^\star_{\mathrm{SP}}$	$D^\star$	$D^\star_{\mathrm{SP}}$	$D^\star$	$D^\star_{\mathrm{SP}}$	$D^\star$	$D^\star_{\mathrm{SP}}$	$D^\star$	$D^\star_{\mathrm{SP}}$	$D^\star$	$D^\star_{\mathrm{SP}}$	$D^\star$	$D^\star_{\mathrm{SP}}$
0.3	DE(1)	3	3	3	3	3	4	4	4	4	6	5	7	7	7	9	8
	EXP(1)	1	1	2	2	3	2	5	5	8	14	12	29	21	59	31	82
	LN(0, 1)	1	1	1	1	2	1	3	2	4	4	6	9	11	21	16	36
	$t(1)$	27	28	32	32	42	42	54	55	65	65	74	74	87	85	93	91
	$t(3)$	5	6	6	7	7	9	10	12	12	15	16	19	21	23	26	27
	U(0, 1)	1	1	1	1	2	2	3	3	5	8	7	18	13	41	18	66
	$\chi^2(1)$	5	6	7	8	13	12	22	37	32	70	45	88	68	99	82	100
	$\chi^2(3)$	1	1	1	1	1	1	2	2	3	4	5	9	9	22	14	41
	$\chi^2(4)$	1	1	1	1	1	1	2	1	2	2	3	4	6	10	8	20
	N(0, 1)	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0.6	DE(1)	6	7	7	8	8	11	11	12	14	15	18	17	27	21	34	24
	EXP(1)	9	9	12	17	16	35	25	62	32	84	44	94	61	100	77	100
	LN(0, 1)	8	7	12	13	15	25	23	45	32	66	41	82	59	95	74	99
	$t(1)$	48	50	57	59	66	69	79	80	86	87	92	92	96	96	96	95
	$t(3)$	7	9	9	12	11	15	14	19	18	22	22	27	28	31	36	36
	U(0, 1)	3	3	3	4	4	9	6	21	8	39	11	59	18	84	26	95
	$\chi^2(1)$	29	42	39	69	49	88	67	99	79	100	89	100	98	100	100	100
	$\chi^2(3)$	5	4	6	7	8	13	12	28	16	47	22	66	34	87	47	97
	$\chi^2(4)$	4	3	4	4	5	8	8	14	10	25	14	39	23	62	30	79
	N(0, 1)	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0.8	DE(1)	6	8	7	9	10	11	13	14	15	17	20	18	27	23	33	27
	EXP(1)	18	31	23	50	33	72	46	92	56	99	70	100	85	100	93	100
	LN(0, 1)	23	35	30	51	41	71	56	89	68	97	80	99	91	100	97	100
	$t(1)$	52	55	60	63	70	72	81	82	87	89	93	93	97	97	99	99
	$t(3)$	8	10	9	12	12	15	15	18	18	23	23	26	28	30	34	36
	U(0, 1)	2	3	2	5	4	11	5	24	6	43	9	59	12	84	17	96
	$\chi^2(1)$	46	79	58	94	73	99	87	100	94	100	98	100	100	100	100	100
	$\chi^2(3)$	9	13	12	23	17	38	26	64	32	83	44	93	58	99	72	100
	$\chi^2(4)$	7	8	8	13	12	22	17	40	20	59	29	73	41	91	52	98
	N(0, 1)	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 7. Estimated power (in %) of the tests based on the indicated statistics, distributions and values of p and n using $\alpha = 0.05$ with parameters estimated by Gupta's method.

		$n = 20$		$n = 25$		$n = 30$		$n = 40$		$n = 50$		$n = 60$		$n = 80$		$n = 100$	
p	Distribution	$D^\star$	$D^\star_{SP}$	$D^\star$	$D^\star_{SP}$	$D^\star$	$D^\star_{SP}$	$D^\star$	$D^\star_{SP}$	$D^\star$	$D^\star_{SP}$	$D^\star$	$D^\star_{SP}$	$D^\star$	$D^\star_{SP}$	$D^\star$	$D^\star_{SP}$
0.3	DE(1)	3	5	2	4	2	6	2	6	2	7	2	9	3	10	8	12
	EXP(1)	21	18	26	24	36	32	47	41	59	53	70	63	82	79	91	91
	LN(0, 1)	16	13	17	15	25	21	33	28	40	33	50	40	62	53	74	65
	$t(1)$	4	21	6	24	13	36	25	49	44	59	65	70	84	83	94	91
	$t(3)$	2	6	2	7	3	10	3	13	5	17	10	22	16	27	28	33
	U(0, 1)	17	14	21	18	29	24	37	31	45	39	56	49	70	64	82	81
	$\chi^2(1)$	44	41	52	50	68	64	83	80	92	90	96	95	99	99	100	100
	$\chi^2(3)$	14	12	16	15	24	20	31	26	38	32	48	39	60	51	73	66
	$\chi^2(4)$	12	10	13	11	18	15	23	18	28	22	36	28	45	36	58	47
	N(0, 1)	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
0.6	DE(1)	6	10	9	12	12	15	17	17	26	24	33	27	46	35	58	42
	EXP(1)	46	41	56	52	67	64	80	81	89	94	93	97	98	100	100	100
	LN(0, 1)	45	40	53	48	63	59	76	74	86	86	92	93	97	98	99	100
	$t(1)$	48	52	59	61	71	71	84	83	92	91	95	95	98	98	100	100
	$t(3)$	10	14	12	17	16	21	21	26	29	32	36	37	44	45	55	52
	U(0, 1)	21	18	27	23	33	30	42	45	53	63	61	77	73	83	83	98
	$\chi^2(1)$	78	75	88	88	94	96	98	100	100	100	100	100	100	100	100	100
	$\chi^2(3)$	32	27	38	34	48	42	60	57	72	73	79	83	89	87	95	99
	$\chi^2(4)$	25	20	30	25	37	32	47	42	55	54	64	64	78	81	86	91
	N(0, 1)	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
0.8	DE(1)	17	16	20	19	24	23	30	27	34	32	39	35	51	42	59	50
	EXP(1)	49	58	62	75	73	88	86	97	93	100	97	100	99	100	100	100
	LN(0, 1)	57	63	69	78	79	88	90	96	95	99	98	100	100	100	100	100
	$t(1)$	64	64	73	73	79	79	88	88	93	93	96	96	99	99	100	100
	$t(3)$	17	18	21	22	23	26	29	31	34	36	38	40	47	47	55	55
	U(0, 1)	12	13	15	19	18	29	24	49	29	67	35	81	49	95	60	99
	$\chi^2(1)$	83	92	92	98	96	100	99	100	100	100	100	100	100	100	100	100
	$\chi^2(3)$	33	39	43	53	51	64	66	83	76	93	85	97	94	100	98	100
	$\chi^2(4)$	24	29	32	40	39	50	52	66	61	79	69	88	84	97	92	99
	N(0, 1)	5	5	5	5	5	6	5	6	5	5	4	5	5	5	5	5

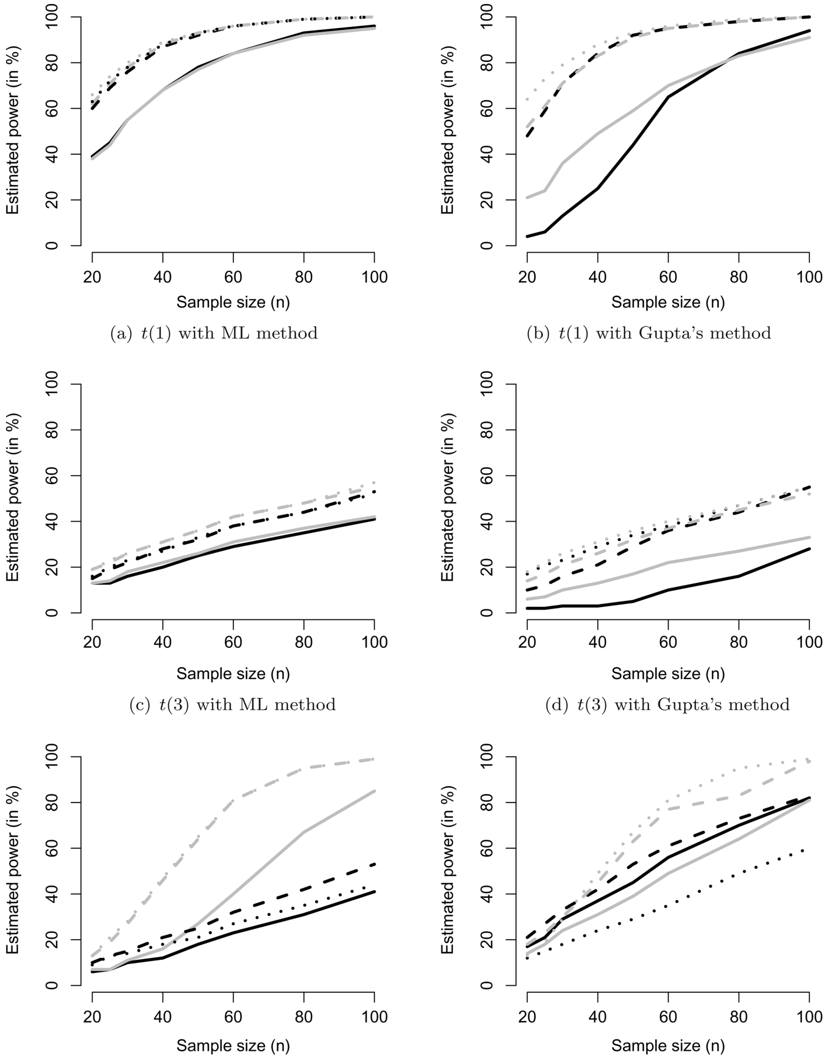


(a) DE(1) with ML method
(b) DE(1) with Gupta's method
(c) EXP(1) with ML method
(d) EXP(1) with Gupta's method
(e) LN(0,1) with ML method
(f) LN(0,1) with Gupta's method

Figure 1. Estimated power of the tests based on $D^\star$ and $p = 0.3$ (bold solid line), $D^\star_{\mathrm{SP}}$ and $p = 0.3$ (gray solid line), $D^\star$ and $p = 0.6$ (bold dashed line), $D^\star_{\mathrm{SP}}$ and $p = 0.6$ (gray dashed line), $D^\star$ and $p = 0.8$ (bold dotted line), and $D^\star_{\mathrm{SP}}$ and $p = 0.8$ (gray dotted line) with parameters estimated using the indicated method for the distribution specified in H_1.

p; see Figures 1(c), 2(e) and 3(e), respectively. As p increases, the power of the test based on $D^\star_{\mathrm{SP}}$ increases too. In particular, it can be noted for $p = 0.8$ and for all the considered values of n. For small values of p and n, both tests are not very powerful, except for the Student-t distribution with 1 degree of freedom; see Tables 5 and 6 and Figure 2(a). For $p > 0.5$, both tests have good power even for $n = 20$, 25.

(3) The test based on $D^\star_{\mathrm{SP}}$ with ML estimation is more powerful than the test based on $D^\star_{\mathrm{SP}}$ with Gupta's estimation, specially with sample sizes greater than 30. On the contrary, the test based on $D^\star$ with ML estimation turns out to be less powerful than the test based on $D^\star$ with Gupta's estimation, except for the DE and Student-t distributions; see Tables 5 and 7 and Figures 1, 2 and 3.



(a) $t(1)$ with ML method

(b) $t(1)$ with Gupta's method

(c) $t(3)$ with ML method

(d) $t(3)$ with Gupta's method

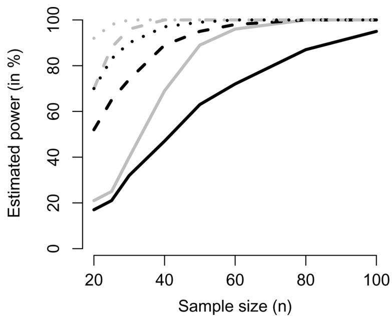
(e) U(0,1) with ML method

(f) U(0,1) with Gupta's method

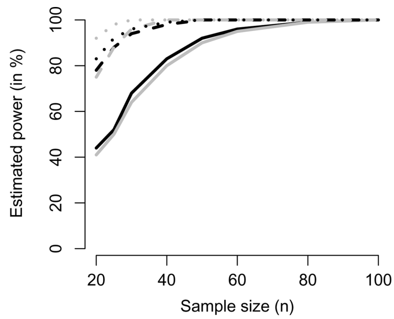
Figure 2. Estimated power of the tests based on $D^\star$ and $p = 0.3$ (bold solid line), $D^\star_{\text{SP}}$ and $p = 0.3$ (gray solid line), $D^\star$ and $p = 0.6$ (bold dashed line), $D^\star_{\text{SP}}$ and $p = 0.6$ (gray dashed line), and $D^\star$ and $p = 0.8$ (bold dotted line), $D^\star_{\text{SP}}$ and $p = 0.8$ (gray dotted line) with parameters estimated using the indicated method for the distribution specified in H_1.

(4) The test based on $D^\star$ is more powerful than the test based on $D^\star_{\text{SP}}$ when the parameters are estimated by Gupta's method for $p = 0.3$, 0.6, specially with sample sizes less than 50, except for the DE and Student-t distributions; see Table 7 and Figures 1(b) and 2(b,d).

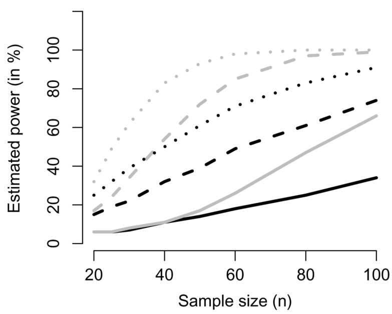
Remark 3.1 Although there are several tests that have very good power, so that they can be recommended for using as omnibus tests, as for example those discussed in [3,4,23], they cannot be associated with graphical procedures. Such procedures, as those based on $D^\star_{\text{SP}}$ and $D^\star$, are frequently suggested because they allow to visualize which data contribute to the decision of rejecting the null hypothesis [13,18].



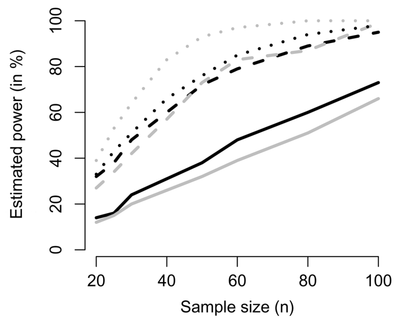
(a) $\chi^2(1)$ with ML method



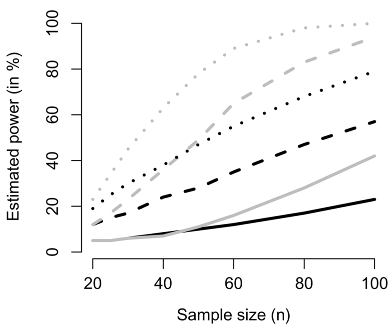
(b) $\chi^2(1)$ with Gupta's method



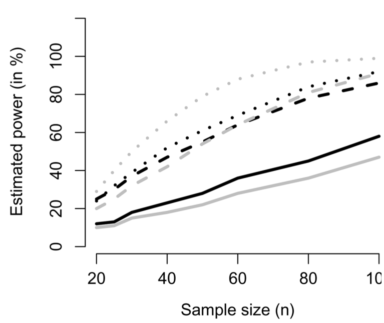
(c) $\chi^2(3)$ with ML method



(d) $\chi^2(3)$ with Gupta's method



(e) $\chi^2(4)$ with ML method



(f) $\chi^2(4)$ with Gupta's method

Figure 3. Estimated power of the tests based on $D^\star$ and $p = 0.3$ (bold solid line), $D^\star_{\mathrm{SP}}$ and $p = 0.3$ (gray solid line), $D^\star$ and $p = 0.6$ (bold dashed line), $D^\star_{\mathrm{SP}}$ and $p = 0.6$ (gray dashed line), and $D^\star$ and $p = 0.8$ (bold dotted line), $D^\star_{\mathrm{SP}}$ and $p = 0.8$ (gray dotted line) with parameters estimated using the indicated method for the distribution specified in H_1.

4. Illustrative example

For the purposes of illustration, we apply the new goodness-of-fit test to a real data set. First, an exploratory data analysis is performed. Then, by using the ML method, the parameters of the normal distribution are estimated considering uncensored and censored data. Finally, by using the proposed goodness-of-fit test, the suitability of the normal model to the data is checked.

The data correspond to life expectancy from birth (in years) of 66 countries. The considered countries had a minimum of 12 million inhabitants in 2004. (These data were obtained from former Table 1318 related to vital statistics provided by the U.S. Census Bureau published in April 2005, now contained in Table 1355. The current link is http://www.allcountries.org/uscensus/1355_vital_statistics_by_country.html, where also other countries appear.) The data

Table 8. Life expectancy (in years) of the indicated country.

Expectancy	Country	Expectancy	Country	Expectancy	Country
81.0	Japan	72.0	China	61.7	Bangladesh
80.3	Australia	72.0	Malaysia	61.4	Yemen
80.0	Canada	71.7	Thailand	60.5	Burma
79.5	Italy	71.4	Colombia	59.4	Nepal
79.4	France	71.4	Brazil	58.6	Cambodia
79.4	Spain	71.1	Romania	58.1	Sudan
78.7	Netherlands	71.1	North Korea	58.1	Ghana
78.5	Germany	70.7	Egypt	56.5	Madagascar
78.3	United Kingdom	70.4	Vietnam	50.7	Congo
77.4	United States	70.4	Morocco	50.7	Cameroon
77.1	Taiwan	69.7	Syria	50.4	Uganda
76.7	South Korea	69.7	Iran	48.7	Ethiopia
76.4	Chile	69.6	Philippines	48.4	Cote d'Ivoire
76.0	Ecuador	69.3	Indonesia	48.0	Burkina Faso
75.7	Argentina	69.2	Peru	47.2	Kenya
75.2	Saudi Arabia	68.8	Ukraine	46.5	Nigeria
74.9	Mexico	68.3	Iraq	44.9	Tanzania
74.7	Poland	66.8	Russia	44.1	South Africa
74.1	Venezuela	66.1	Kazakhstan	42.5	Afghanistan
72.9	Sri Lanka	64.1	Uzbekistan	41.2	Malawi
72.7	Algeria	64.0	India	40.9	Mozambique
72.1	Turkey	62.6	Pakistan	39.0	Zimbabwe

Table 9. Descriptive statistics for the life expectancy (in years) of the uncensored data.

Mean	Median	SD	CV	CS	CK	Range	Min.	Max.	n
65.6	69.7	11.9	18.2	−0.725	−0.739	42	39	81	66

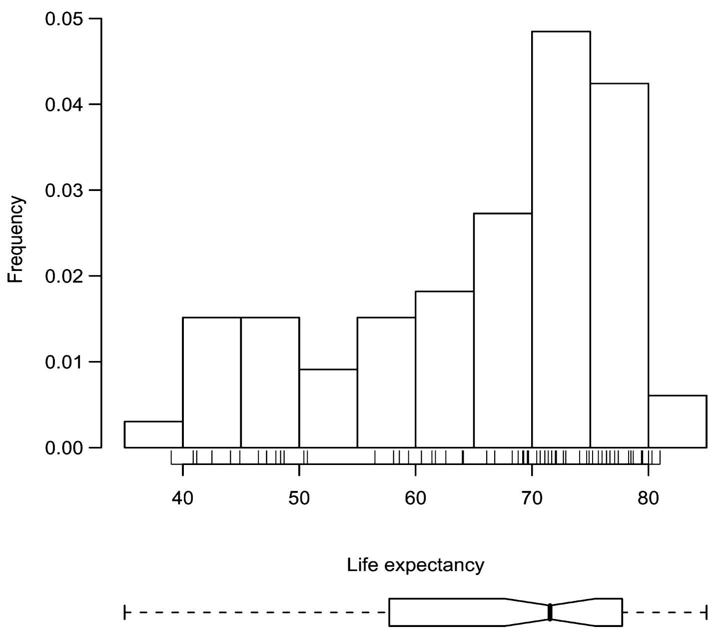
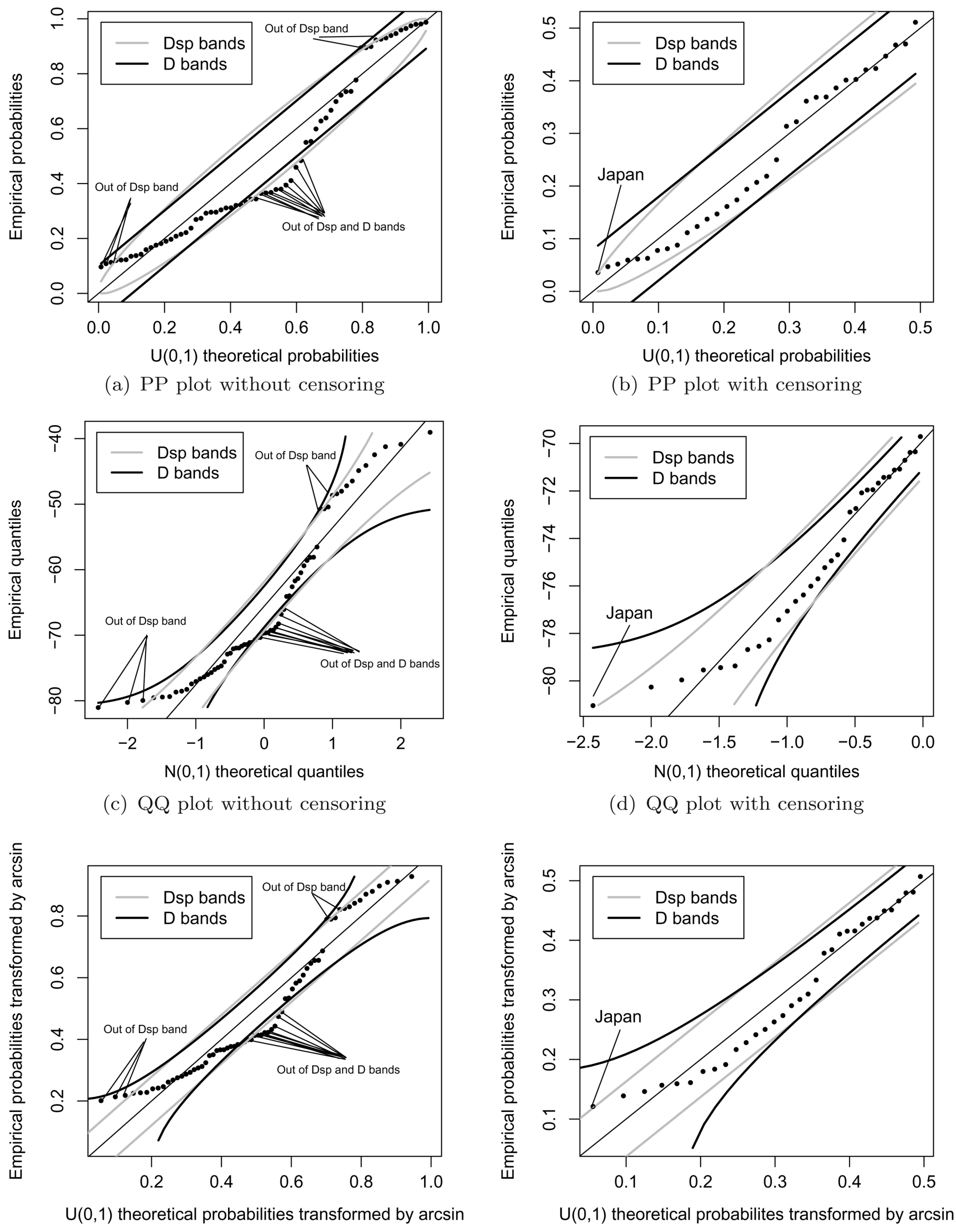


Figure 4. Histogram and boxplot for the life expectancy (in years) of the uncensored data.

in Table 8 are displayed in decreasing order with respect to the life expectancy of each country.

Table 9 presents a descriptive summary of the $n = 66$ observations of the uncensored sample, while Figure 4 shows the corresponding histogram and boxplot from which it is possible to note that the normal distribution is not a good model for describing these data. The proposed goodness-of-fit test should confirm this fact. From the original data, a left type-II censored sample was generated



(a) PP plot without censoring
(b) PP plot with censoring
(c) QQ plot without censoring
(d) QQ plot with censoring
(e) SP plot without censoring
(f) SP plot with censoring

Figure 5. 95% acceptance regions based on $D^{\star}$ and $D^{\star}_{\mathrm{SP}}$ on the indicated probability plots with and without censoring.

for $p = 0.5$. A right type-II censored sample was obtained changing the sign of the data. Thus, we have $r = 33$ countries with uncensored life expectancy in our sample of size $n = 66$, with the highest life expectancy in 2004 and $n - r = 33$ countries with censored life expectancy. The ML estimates of μ and σ for the uncensored and 50% censored samples are $\hat{\mu} = 65.6$ and $\hat{\sigma} = 11.9$, and $\hat{\mu}_c = 69.9$ and $\hat{\sigma}_c = 6.2$ years old.

For the uncensored sample (using negative values for the life expectancy) Figures 5a–c give the PP, SP and QQ plots, respectively, with their corresponding 95% acceptance regions. From these plots, we can confirm that the data do not follow a normal distribution, since several observations fall outside the 95% acceptance regions derived from $D^\star$ and $D^\star_{SP}$. The observed values of $D^\star$ and $D^\star_{SP}$ for H_0: "the distribution is $N(\mu, \sigma^2)$" are: $d^\star = 0.1815$ (p-value < 0.01) and $d^\star_{SP} = 0.1457$ (p-value < 0.01); see Table 10. As expected, both tests reject the null hypothesis of normality at $\alpha = 0.01$. We can point out that in the case of the acceptance regions derived from $D^\star_{SP}$, the observations falling out of these regions correspond to Japan, Australia, Canada, Morocco, Syria, Iran, Philippines, Indonesia, Peru, Ukraine, Iraq, Russia, Kazakhstan, Congo, and Ethiopia. In the case of the acceptance regions based on $D^\star$, the observations falling out of these regions correspond to North Korea, Egypt, Vietnam, Morocco, Syria, Iran, Philippines, Indonesia, Peru, Ukraine, Iraq, Russia, and Kazakhstan. Furthermore, we can note that the points in the graph do not tend to lie on a straight line, which indicates a bad specification of the postulated hypothetical distribution, in this case, the normal model.

Remark 4.1 In the PP plots, the straight lines correspond to the acceptance regions of the test based on $D^\star$ and the curves to the ones based on $D^\star_{SP}$. For the SP plots, the straight lines correspond to the acceptance regions of the test based on $D^\star_{SP}$ and the curves to the ones based on $D^\star$. These aspects can be corroborated in Table 4 and, in the case of the example data, in Figure 5a,b,e, and f.

For the censored sample (using negative values for the life expectancy) Figure 5d–f give the QQ, PP and SP plots, respectively, with their corresponding 95% acceptance regions. From these plots, based on $D^\star$, we have no evidence to indicate that the data do not follow a normal distribution, because all the observations fall inside the 95% acceptance regions. However, for the 95% acceptance regions based on $D^\star_{SP}$, there is one point falling out of these regions, which corresponds to Japan with a life expectancy of 81 years old. Note that the conclusions using both statistics differ. Although the true distribution of the uncensored sample is unknown, based on the evidence of the whole sample, we detect that this model does not correspond to the normal one and hence the proposed $D^\star_{SP}$ leads to a more adequate conclusion. The observed values of $D^\star$ and $D^\star_{SP}$ for H_0: "the distribution is $N(\mu,\sigma^2)$" based on the 50% censored sample are: $d^\star = 0.054$ ($0.5 < p$-value < 0.6) and $d^\star_{SP} = 0.066$ ($0.01 < p$-value < 0.05); see Table 10. Therefore, the test based on $D^\star_{SP}$ rejects H_0 at the 5% level of significance, but the test based on $D^\star$ does not. This is coherent with the observed behavior of the empirical power of both tests.

Table 10. Quantiles of the distributions of $D^\star$ and $D^\star_{SP}$ for a normal distribution under H_0 with parameters estimated by ML method and right type II censoring for the indicated values of p, n, and $1 - \alpha$.

p	n	$d^\star(0.50)$	$d^\star(0.75)$	$d^\star(0.90)$	$d^\star(0.95)$	$d^\star(0.99)$
0.50	66	0.0571	0.0678	0.0795	0.0870	0.1018
		$d^\star_{SP}(0.50)$	$d^\star_{SP}(0.75)$	$d^\star_{SP}(0.90)$	$d^\star_{SP}(0.95)$	$d^\star_{SP}(0.99)$
0.50	66	0.0416	0.0498	0.0584	0.0643	0.0755

Conclusions

In this paper, we have proposed a new goodness-of-fit test for a normal distribution with unknown parameters and right type-II censored data. In addition, we have compared this test to a modified Kolmogorov–Smirnov test for censored data. The new test is more powerful than the Kolmogorov–Smirnov test in most of the cases studied. Both tests can also be applied to left type-II censoring and to the LN distribution. One advantage of the proposed tests is that they offer the possibility of drawing acceptance regions on probability plots, where not only the rejection or acceptance of the null hypothesis can be established, but also the points that make that decision. These plots can only be obtained for tests based on distances from the empirical distribution function, while other tests for normality with unknown parameters have acceptance regions that cannot be drawn on probability plots. The authors are developing an `R` [21] package to make the obtained results available on CRAN (http://CRAN.R-project.org/) allowing practitioners to use the proposed tests.

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