

RESEARCH ARTICLE

Robustness properties of an algorithm for the stabilization of switched systems with unbounded perturbations

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In this paper it is shown that an algorithm for the stabilization of switched systems introduced by Mancilla-Aguilar and García (Automatica, 49 (2013) 441-447) is robust with respect to perturbations which are unbounded in the supremum norm, but bounded in a power-like sense. The obtained stability results comprise, among others, both the exponential input-to-state stability (ISS) and the exponential integral input-to-state stability (iISS) properties of the closed-loop system and give a better description of the behavior of the closed-loop system.

Keywords: Switched systems, Robust stabilization, ISS, Exponential stabilization.

1. Introduction

In the last years switched systems turned out to be a well established area of research in control, both in theory and applications, mainly because they allow one to describe the behaviour of a large class of plants resulting from the interactions of continuous dynamics, discrete dynamics, and logic decisions (Liberzon, 2003). Many instances of mechanical, electric power and control systems can be modeled as switched systems (Liberzon, 2003; Liberzon and Morse, 1999; Matveev and Savkin, 2000; van der Schaft and Schumacher, 2000). Informally, a switched system is a family of continuous-time dynamical subsystems and a usually time-dependent or state-dependent law—the *switching signal*—that rules the switching between the subsystems. Although these systems may look simple, their behavior may be very complicated. The stability properties of switched systems for instance, may radically differ from those of their component dynamical subsystems. In fact, switching between two stable linear time-invariant subsystems may result in unstable behavior, while switching between two unstable linear time-invariant subsystems may yield stability (De Carlo et al., 2000). Consequently, many efforts have been devoted to the study of the different stability properties of switched systems (see De Carlo et al., 2000; Liberzon, 2003; Liberzon and Morse, 1999; Lin and Antsaklis, 2009; Shorten et al., 2007, and references therein). Another important problem in the theory of switched system is the so-called switching stabilization problem (Liberzon and Morse, 1999), that may be stated as follows:

Construct switching signals that make the origin an asymptotically stable point of the switched system.

In the case of continuous-time switched linear systems different solutions to this problem were presented in the literature, either with state-feedback switching laws that stabilize the switched system (see Bacciotti, 2004; Lin and Antsaklis, 2009; Peleties et al., 1994) or with mixed time-driven (*i. e.* open-loop) and state-feedback switching mechanisms (Sun, 2006, 2012; Sun and Ge,

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2011). Recent developments for discrete-time switched linear systems can be found in (Hu et al., 2009; Lee and Khargonekar, 2009; Sun, 2009).

In the construction of state-feedback stabilizing switching laws for nonlinear systems, a weak (or control) Lyapunov function or a family of them (see Bacciotti, 2004; Liberzon, 2003; Liu et al., 2010, and references therein) is usually employed, and the switching signal is implemented by using some kind of hysteresis in order to avoid both Zeno behavior and chattering. For time-driven stabilizing switching signals some results can be found in (Bacciotti and Mazzi, 2010) and in (Mancilla-Aguilar and García, 2013) where the stabilization is with respect to compact sets and in a practical sense. It must be pointed out, that in this case, the design of the switching signal does not rely on the knowledge of Lyapunov functions of any kind.

Closely related to the stabilization problem is that of the robustness of the stabilizing switching laws, that is, to seek proper switching mechanisms that make the system stable and attenuate possible system disturbances or perturbations. Design schemes for robust switching of switched linear systems when the design is based on control Lyapunov functions, were presented in (Sun, 2009, 2012; Xie and Wang, 2005; Xu et al., 2007). In (Mancilla-Aguilar and García, 2015) it was shown that for continuous-time switched systems whose subsystems are linear, or, more generally, *homogeneous of degree one* (see the definition below), the design presented in (Mancilla-Aguilar and García, 2013) exponentially stabilizes the switched system in a practical sense, with a final error which depends linearly on the bounds of both the model uncertainties and the measurement errors. In other words, the closed-loop system is exponentially input-to-state stable (exponentially ISS) if the perturbation in the system model and the measurement error are seen as inputs.

Although the ISS property is important and useful for a wide range of control problems (see Khalil, 2002), it gives no robustness information in the case in which the inputs have bounded power or energy but are unbounded. This fact motivates the introduction of different characterizations of robustness (see Angeli and Nešić, 2001). In this paper we consider the case in which the perturbations are bounded in a novel, compared to previous characterizations, power-like sense. One of the contributions of the paper is to show that the design presented in (Mancilla-Aguilar and García, 2013) semi-globally stabilizes the perturbed switched system in a practical sense, and that the stability is robust with respect to small errors in the measurements and to small (in a power-like integral norm) perturbations. Another contribution of the paper is, in the case of switched systems whose subsystems are *homogeneous of degree one*, to prove for this controller robust exponential stability results in the case of locally integrable perturbations. It must be pointed out that this general result comprises, among others, the exponential ISS and exponential iISS (integral input-to-state-stability, (see Sontag, 1998)) properties of that design, but gives us a better description of the closed-loop system.

The paper is organized as follows. In section 2 we give the basic definitions, embed the switched system into a control-affine nonlinear one and recall the stabilizer algorithm for the control-affine systems presented in (Mancilla-Aguilar and García, 2013). In section 3 we prove for this stabilizer robustness results for locally integrable perturbations, and obtain robust exponential stability results for switched homogeneous systems with this type of perturbations. In section 4 we illustrate the obtained results by means of an example while in section 5 we present some conclusions.

2. Preliminaries

In the sequel $\langle \cdot, \cdot \rangle$ is the standard inner product on \mathbb{R}^n and $|\xi| = \langle \xi, \xi \rangle^{1/2}$ is the Euclidean norm of $\xi \in \mathbb{R}^n$. For a nonempty set $A \subset \mathbb{R}^n$ and for any $x \in \mathbb{R}^n$, $|x|_A = \inf_{a \in A} |x - a|$ is the distance from x to A . Given a subset $C \subset \mathbb{R}^N$, $\text{co}(C)$ denotes its convex hull. Given $s \in \mathbb{R}$, $\lfloor s \rfloor$ stands for the greatest integer number less or equal to s . For any $m \in \mathbb{N}$, L^∞ and $L^\infty_{m,loc}$ denote the sets of all the locally essentially bounded, and respectively the set of all the essentially bounded functions $u : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$. With $\|u\|_\infty = \text{ess sup}_{t \geq 0} |u(t)|$ we denote the essential supremum norm of $u \in L^\infty$.

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is homogeneous of degree one if $f(\lambda x) = \lambda f(x)$ for all $x \in \mathbb{R}^n$ and for all $\lambda > 0$. We note that an homogeneous of degree one map which is locally Lipschitz is indeed globally Lipschitz.

In this work we will consider the switched system

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \quad (1)$$

where $\sigma : [0, \infty) \rightarrow \{1, \dots, N\}$ is the switching signal, i.e. σ is a piecewise constant and continuous from the right function, and for each $i \in \{1, \dots, N\}$, $f_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is locally Lipschitz

As in (Mancilla-Aguilar and García, 2013) we embed system (1) into the control-affine system

$$\dot{z}(t) = \sum_{i=1}^N u_i(t) f_i(z(t)) := F(z(t))u(t) \quad (2)$$

where $F(z) = [f_1(z) \dots f_N(z)] \in \mathbb{R}^{n \times N}$ and for $t \geq 0$, $z(t) \in \mathbb{R}^n$ and $u(t) \in U = \text{co}(U^*)$, with $U^* = \{e_1, \dots, e_N\}$, where $e_i \in \mathbb{R}^N$ denotes the i -th canonical vector of \mathbb{R}^N . We assume that the admissible controls of (2) belong to \mathcal{U} , the set of all the Lebesgue measurable functions $u : [0, \infty) \rightarrow U$.

The embedding of (1) into (2) is performed by identifying the set \mathcal{S} of all the switching signals of (1) with the set \mathcal{U}_{pc}^* of all controls $u \in \mathcal{U}$ that take values in U^* and are piecewise constant and continuous from the right, by means of the bijection $\sigma \mapsto u_\sigma$, $u_\sigma(\cdot) = e_{\sigma(\cdot)}$.

Remark 1: We note that the trajectories of (1) corresponding to a switching signal σ are the same as those of (2) which correspond to the control u_σ . In consequence, in the sequel we will identify the trajectories of the switched system with those of the control system corresponding to piecewise constant controls.

For $z_0 \in \mathbb{R}^n$ and $u \in \mathcal{U}$, we denote by $z(\cdot, z_0, u)$ the unique maximally defined solution of (2) which verifies $z(0, z_0, u) = z_0$, and by $I_{z_0, u} = [0, t_{z_0, u})$ its interval of definition.

Next we recall some definitions introduced in Mancilla-Aguilar and García (2013) and Mancilla-Aguilar and García (2015).

Definition 1: The control system (2) is \mathcal{U} -stabilizable if there exists a parametrized family $\Sigma = \{u_{z_0}\}_{z_0 \in \mathbb{R}^n}$ of controls in \mathcal{U} such that for some function $\beta \in \mathcal{KL}^1$ and for all $z_0 \in \mathbb{R}^n \geq 0$ and all $t \geq 0$,

$$|z(t, z_0, u_{z_0})| \leq \beta(|z_0|, t).$$

In the sequel, Σ will be referred to as a \mathcal{U} -stabilizer of (2).

Definition 2: A parametrized family of controls $\Sigma = \{u_{z_0}\}_{z_0 \in \mathbb{R}^n}$ is scale-invariant if for any $z_0 \neq 0$, $u_{z_0} = u_{\tilde{z}_0}$ with $\tilde{z}_0 = \frac{z_0}{\|z_0\|}$.

Remark 2: We note that the controls in a \mathcal{U} -stabilizer Σ asymptotically drive the states of the control system to the origin. We also note that they are not necessarily switching signals.

¹As usual, by a \mathcal{K}_∞ -function we mean a continuous function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ that is strictly increasing and unbounded, and satisfies $\alpha(0) = 0$ and by \mathcal{KL} the set of functions $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ that are of class \mathcal{K}_∞ in the first argument and decrease to 0 when the second argument goes to ∞ .

2.1 Control algorithm

Here we recall the control algorithm $\mathcal{C}(T, \tau, \Sigma)$ introduced in (Mancilla-Aguilar and García, 2013) for the stabilization of switched systems. In order to take into account external disturbances, model uncertainties, and measurement errors in the analysis of the closed-loop system, instead of (2), we consider the perturbed control system

$$\dot{z}(t) = F(z(t))u(t) + \rho(t), \quad (3)$$

where the *perturbation* $\rho : [0, \infty) \rightarrow \mathbb{R}_{\geq 0}$ is a Lebesgue measurable and locally essentially bounded function. We also assume that a measurement $\hat{z}(t)$ of the state $z(t)$ is available for all $t \geq 0$, and that $\hat{z}(\cdot)$ satisfies

$$|\hat{z}(t) - z(t)| \leq \rho_2 \quad \forall t \geq 0, \quad (4)$$

for some positive number ρ_2 . In the sequel we will denote by ϵ the *measurement error*, i.e. $\epsilon(\cdot) = \hat{z}(\cdot) - z(\cdot)$.

Controller $\mathcal{C}(T, \tau, \Sigma)$.

Let $T > 0$ be a tracking period, $\tau > 0$ be a dwell-time such that $T = k\tau$ with $k \in \mathbb{N}$, and let $\Sigma = \{u_{z_0}\}_{z_0 \in \mathbb{R}^n}$ be a \mathcal{U} -stabilizer of (2). Then, the stabilizer controller $\mathcal{C}(T, \tau, \Sigma)$ generates the control $u(\cdot) \in \mathcal{U}_{pc}^*$ to be applied to (3) as follows:

Initialization

$j := 0, T_0 := 0$

Recursive step

(*) $T_{j+1} := T_j + T$ and $u(t) = u_j(t)$ for all $t \in [T_j, T_{j+1})$, where $u_j(t)$ is the control signal provided by the *tracking controller* $\mathcal{T}(T, \tau, \xi_j)$ with $\xi_j : [T_j, T_{j+1}) \rightarrow \mathbb{R}^n$ defined by

$$\xi_j(t) = z(t - T_j, \hat{z}(T_j), u_{\hat{z}(T_j)}); \quad (5)$$

$j := j + 1$. Go to (*).

Tracking controller $\mathcal{T}(T, \tau, \xi)$.

Given a tracking period $T > 0$, a dwell-time $\tau > 0$, and a trajectory of (2) $\xi : [s, s + T) \rightarrow \mathbb{R}^n$, with $s \geq 0$, which is generated by some control $v \in \mathcal{U}$ and which is to be tracked by (3), the tracking controller $\mathcal{T}(T, \tau, \xi)$ generates the control $u : [s, s + T) \rightarrow U^*$ to be applied to (3) as follows.

Let $t_i = s + i\tau$, $i = 0, \dots, \lfloor \frac{T}{\tau} \rfloor$ be the switching times of u . Then the control $u(t)$ is defined by the rule:

$$\begin{aligned} u(t) &= u(t_i) \text{ on } [t_i, t_i + \tau) \cap [s, s + T), \\ u(t_i) &\in \varphi(\xi(t_i), \hat{z}(t_i)). \end{aligned}$$

where the set-valued map $\varphi : \mathbb{R}^n \times \mathbb{R}^n \rightsquigarrow U^*$ is defined by

$$\varphi(\xi, z) = \arg \max_{v \in U^*} \langle \xi - z, F(z)v \rangle.$$

Remark 3: The controller $\mathcal{C}(T, \tau, \mu, \Sigma)$ has three components: a) a generator of references which are obtained from an internal model of the control system controlled by Σ , b) a tracking algorithm which provides to the system the control signal for tracking the reference given by the generator of references and c) a supervisor that periodically updates the reference to be tracked.

Remark 4: We note that if $u \in U^*$ belongs to $\varphi(\xi, z)$, then $u = e_k$, with $k \in \{1, \dots, N\}$ such that

$$\langle \xi - z, f_k(z) \rangle = \max_{1 \leq j \leq N} \langle \xi - z, f_j(z) \rangle.$$

For such u it also holds that

$$\langle \xi - z, F(z)u \rangle = \max_{v \in U} \langle \xi - z, F(z)v \rangle.$$

Remark 5: The controller $\mathcal{C}(T, \tau, \Sigma)$ generates a piecewise control u which can be converted into a switching signal σ by means of the inverse of the bijection described above, that is, for any $t \geq 0$, $\sigma(t) = k$ if and only if $u(t) = e_k$. By construction such a σ has dwell time τ .

3. Robustness results for integrable perturbations

In previous works it was proven that by a proper selection of the parameters T and τ the control algorithm $\mathcal{C}(T, \tau, \Sigma)$ semiglobally stabilizes the perturbed control system (3) in a practical sense (see Theorem 11 in (Mancilla-Aguilar and García, 2013)), provided that the magnitudes of the perturbation $\rho(\cdot)$ and of the bound ρ_2 of the measurement error $\epsilon(\cdot)$ are small enough. In (Mancilla-Aguilar and García, 2015), it was shown that in the case in which the maps f_i are homogeneous of degree one and Σ is a scale-invariant \mathcal{U} -stabilizer of (2), the controller $\mathcal{C}(T, \tau, \Sigma)$ with properly selected parameters T and τ , exponentially stabilizes the unperturbed system (2) and renders the perturbed system (3) ultimately bounded when both the perturbation $\rho(\cdot)$ and the measurement error $\epsilon(\cdot)$ are bounded; more precisely, the closed-loop system is exponentially input-to-state stable (exponentially ISS) if the perturbation $\rho(\cdot)$ and the measurement error $\epsilon(\cdot)$ are seen as inputs. Although ISS is a very interesting and useful property, it cannot deal with perturbations (or inputs) which are unbounded. Neither does it give a good description of the behaviour of the system when the perturbations take occasionally values of very high magnitude but have uniformly bounded energy on intervals of fixed length, for example, when the perturbation is a train of equally spaced pulses of high magnitude, short duration and uniformly bounded energy.

In order to analyze the effect of perturbations ρ like those mentioned above in the behaviour of the control system (3) controlled by $\mathcal{C}(T, \tau, \Sigma)$, we introduce the following family of norms, which we will collectively name the *power norms*. Given $p \in [1, \infty)$ and $T > 0$, we define for $\rho \in L_{m,loc}^\infty$:

$$\|\rho\|_{p,T} := \sup_{t \geq 0} \left(\int_t^{t+T} |\rho(s)|^p ds \right)^{1/p}. \quad (6)$$

Remark 6: The following assertions hold.

- (1) Given positive real numbers T and T' , there exists a constant $k = k(T, T')$ such that $\|\rho\|_{1,T} \leq k\|\rho\|_{1,T'}$. In consequence, if $\|\rho\|_{1,T}$ is finite for some $T > 0$, then $\|\rho\|_{1,T'}$ is finite for every $T' > 0$.
- (2) Taking into account that for $p > 1$, $T > 0$ and every $t \geq 0$, $\int_t^{t+T} |\rho(s)| ds \leq T^{1/q} (\int_t^{t+T} |\rho(s)|^p ds)^{1/p}$, where q is the conjugate exponent of p (i.e. $1/p + 1/q = 1$), we have that $\|\rho\|_{1,T} \leq T^{1/q} \|\rho\|_{p,T}$.
- (3) Since for every $T > 0$ and every $t \geq 0$, $\int_t^{t+T} |\rho(s)| ds \leq T (\text{ess.sup.}\{|\rho(s)| : t \leq s \leq t+T\}) \leq T \|\rho\|_\infty$, it follows that $\|\rho\|_{1,T} \leq T \|\rho\|_\infty$.

3.1 General switched systems

First we study the behaviour of the perturbed control system (3) when it is controlled by $\mathcal{C}(T, \tau, \Sigma)$ and $\|\rho\|_{1,T}$ is finite for any $T > 0$, without introducing further hypotheses.

In order to do so we introduce the following constants. For a given non-empty compact set $K \subset \mathbb{R}^n$ let $l = l_K$ and $m = m_K$ be such that

$$|f_i(x) - f_i(z)| \leq l|x - z|, \quad \forall x, z \in K, \quad \forall i \in \{1, \dots, N\}, \quad (7)$$

and

$$|f_i(x)| \leq m, \quad \forall x \in K, \quad \forall i \in \{1, \dots, N\}. \quad (8)$$

We also define the constants Λ_i , $i = 1, 2$, as follows

$$\Lambda_1 = 4(l + m) \quad \text{and} \quad \Lambda_2 = 4lm + 8m^2.$$

Finally, for nonnegative real numbers t , τ , ρ_1 and ρ_2 let

$$\Gamma_K(t, \tau, \rho_1, \rho_2) = \left[(\rho_2^2 + \rho_1)e^{2lt} + \frac{e^{2lt} - 1}{2l}(\Lambda_1(\rho_1 + \rho_2) + \Lambda_2\tau) \right]^{1/2}.$$

We note that $\Gamma_K(\cdot, \tau, \rho_1, \rho_2)$ is strictly increasing and that $\Gamma_K(t, \tau, \rho_1, \rho_2) \rightarrow 0$ as $\tau + \rho_1 + \rho_2 \rightarrow 0$, uniformly when t varies on a compact set of $[0, \infty)$.

Lemma 8 in (Mancilla-Aguilar and García, 2013) shows that, for a fixed $T > 0$, the tracking controller $\mathcal{T}(T, \tau, \xi)$ previously described is robust with respect to small measurement errors and perturbations which are small in magnitude (*i.e.* in the supremum norm). Since the system is modelled by a differential equation (see equation (3)), the interval under analysis has length T and the perturbation is additive and affects the state via its derivative, what really matters is the integral of the absolute value of the perturbation on that interval. So, it is plausible that a result analogous to the lemma mentioned above holds with a power norm of the perturbation instead of the supremum one. This result is stated in the following lemma.

Lemma 1: Let $\xi : [s, s + T] \rightarrow \mathbb{R}^n$, with $s \geq 0$ and $T > 0$, be a trajectory of (2) corresponding to a control $w \in \mathcal{U}$. Let $K \subset \mathbb{R}^n$ be a compact set such that $|\xi(t)|_{\mathbb{R}^n \setminus K} \geq 2$ for all $t \in [s, s + T]$. Assume that $\tau > 0$, $\rho_1 > 0$ and $\rho_2 > 0$ verify $\Gamma_K(T, \tau, \rho_1, \rho_2) < 1$.

Then, if $z(\cdot)$ is a trajectory of (3) controlled by $\mathcal{T}(T, \tau, \xi)$ such that $\rho(\cdot)$ in (3) satisfies $\|\rho\|_{1,T} \leq \rho_1$, $|\epsilon(t)| \leq \rho_2$ for all $t \in [s, s + T]$ and $|z(s) - \xi(s)| \leq \rho_2$, we have that

$$|z(t) - \xi(t)| \leq \Gamma_K(t - s, \tau, \rho_1, \rho_2) \quad \forall t \in [s, s + T]. \quad (9)$$

Proof. Let $z(\cdot)$ be a trajectory of (3) controlled by $\mathcal{T}(T, \tau, \xi)$ and let $\mu(t) = z(t) - \xi(t)$ for all $t \in [s, s + T]$. Assume that the perturbation $\rho(\cdot)$ in (3) satisfies $\|\rho\|_{1,T} \leq \rho_1$, that $|\epsilon(t)| \leq \rho_2$ for all $t \in [s, s + T]$ and that $|\mu(s)| \leq \rho_2$. Also suppose that $\Gamma_K(T, \tau, \rho_1, \rho_2) < 1$.

Let $I = \{\hat{t} \in [s, s + T] : |\mu(\hat{t})| \leq 1 \quad \forall t \in [s, \hat{t}]\}$. Then I is a closed interval, *i.e.* $I = [s, t^z]$ and $t^z > s$ since $|\mu(s)| \leq \rho_2 \leq \Gamma_{K,U}(T, \tau, \rho_1, \rho_2) < 1$. The lemma follows if we prove that (9) holds for all $t \in I$, since in that case $t^z = s + T$. In fact, if $t^z < s + T$, it follows from (9) that $|\mu(t^z)| \leq \Gamma_{K,U}(T, \tau, \rho_1, \rho_2) < 1$; given that $\mu(\cdot)$ is a continuous function, there exists $t' > t^z$ such that $|\mu(t)| \leq 1$ for all $t^z \leq t \leq t'$ and, in consequence, $[s, t'] \subseteq I$, which is a contradiction.

Let $r(t) = |\mu(t)|^2$ for all $t \in [s, s + T]$. Then $r(\cdot)$ is absolutely continuous on I and, for almost all

$t \in I$,

$$\begin{aligned}
\dot{r}(t) &\leq 2\langle \mu(t), F(z(t))u(t) \rangle + \rho(t) - F(\xi(t))w(t) \\
&\leq 2|\mu(t)||\rho(t)| + 2\langle \mu(t), F(z(t))u(t) - F(\xi(t))w(t) \rangle \\
&\leq 2|\rho(t)| + 2\langle \mu(t), F(z(t))u(t) - F(\xi(t))w(t) \rangle,
\end{aligned} \tag{10}$$

where $u(\cdot)$ is the control provided by the controller $\mathcal{T}(T, \tau, \xi)$.

In order to bound $c(t) = \langle \mu(t), F(z(t))u(t) - F(\xi(t))w(t) \rangle$, we write

$$c(t) = c_1(t) + c_2(t) + c_3(t) + c_4(t) + c_5(t), \tag{11}$$

where

$$\begin{aligned}
c_1(t) &= \langle \mu(t), F(z(t))w(t) - F(\xi(t))w(t) \rangle \\
c_2(t) &= \langle \mu(t), F(z(t))u(t) - F(\hat{z}(t_k))u(t) + F(\hat{z}(t_k))w(t) - F(z(t))w(t) \rangle \\
c_3(t) &= \langle \mu(t) - \mu(t_k), F(\hat{z}(t_k))u(t) - F(\hat{z}(t_k))w(t) \rangle \\
c_4(t) &= \langle z(t_k) - \hat{z}(t_k), F(\hat{z}(t_k))u(t) - F(\hat{z}(t_k))w(t) \rangle \\
c_5(t) &= \langle \hat{z}(t_k) - \xi(t_k), F(\hat{z}(t_k))u(t) - F(\hat{z}(t_k))w(t) \rangle,
\end{aligned}$$

and $t_k = s + k\tau$ is such that $t_k \leq t < t_{k+1}$.

Since for all $t \in I$, $|\xi(t)|_{\mathbb{R}^n \setminus K} \geq 2$, $|\mu(t)| \leq 1$ and $|\epsilon(t)| \leq \rho_2 \leq \Gamma_K(T, \tau, \rho_1, \rho_2) < 1$, it follows that $z(t) \in K$ and $\hat{z}(t) \in K$ for all $t \in I$.

Then, from (7)

$$c_1(t) \leq l|\mu(t)||\mu(t)| = lr(t), \quad \forall t \in I. \tag{12}$$

Taking into account (7), (8) and that for all $t \in I$

$$\begin{aligned}
|z(t) - z(t_k)| &\leq \int_{t_k}^t (|f(z(s), u(s))| + |\rho(s)|) ds \\
&\leq m\tau + \|\rho\|_{1,T} \leq m\tau + \rho_1, \\
|z(t) - \hat{z}(t_k)| &\leq |z(t) - z(t_k)| + |z(t_k) - \hat{z}(t_k)| \\
&\leq m\tau + \rho_1 + \rho_2,
\end{aligned}$$

and

$$|\xi(t) - \xi(t_k)| \leq \int_{t_k}^t |f(\xi(s), w(s))| ds \leq m\tau,$$

it follows that for all $t \in I$

$$\begin{aligned}
c_2(t) &\leq 2|\mu(t)|l|z(t) - \hat{z}(t_k)| \\
&\leq 2l[m\tau + \rho_1 + \rho_2],
\end{aligned} \tag{13}$$

$$\begin{aligned}
c_3(t) &\leq 2|\mu(t) - \mu(t_k)|m \\
&\leq 2m(2m\tau + \rho_1),
\end{aligned} \tag{14}$$

$$c_4(t) \leq 2m\rho_2. \tag{15}$$

Finally, from the fact that $u(t_k) \in \varphi(\xi(t_k), \hat{z}(t_k))$ and from Remark 4, we have that

$$c_5(t) \leq 0 \quad \forall t \in I. \quad (16)$$

Therefore, from (11)-(16),

$$c(t) \leq lr(t) + 2l[m\tau + \rho_1 + \rho_2] + 2m(2m\tau + \rho_1) + 2m\rho_2,$$

and, in consequence,

$$\dot{r}(t) \leq 2lr(t) + 2|\rho(t)| + \Lambda_1\rho_1 + \Lambda_2\rho_2 + \Lambda_3\tau$$

for almost all $t \in I$. Then Gronwall-Bellman Lemma yields

$$\begin{aligned} r(t) &\leq r(s)e^{2l(t-s)} + \int_s^t e^{2l(t-y)} |\rho(y)| dy + \frac{(e^{2l(t-s)} - 1)}{2l} [\Lambda_1(\rho_1 + \rho_2) + \Lambda_2\tau] \\ &\leq (\rho_2^2 + \rho_1)e^{2l(t-s)} + \frac{(e^{2l(t-s)} - 1)}{2l} [\Lambda_1(\rho_1 + \rho_2) + \Lambda_2\tau] \\ &= (\Gamma(t-s, \tau, \rho_1, \rho_2))^2, \end{aligned}$$

or, equivalently,

$$|\mu(t)| \leq \Gamma(t-s, \tau, \rho_1, \rho_2) \quad \forall t \in I.$$

□

The next result, which is one of the main ones of the paper, can be obtained following the same steps that those used in the proof Theorem 11 in (Mancilla-Aguilar and García, 2013), but using Lemma 1 instead of Lemma 8 in that paper. It shows that the family of controllers $\mathcal{C}(T, \tau, \Sigma)$ semi-globally stabilizes the system (2) in a practical sense, and that the stability is robust with respect to small errors in the measurements and small (in a power norm) perturbations.

Theorem 1: Let Σ a \mathcal{U} -stabilizer of (2) and $0 < \varepsilon_0 < R_0$. Then there exist positive numbers T , τ_0 , ρ_1^* and ρ_2^* , such that if the perturbation $\rho(\cdot)$ in (3) verifies $\|\rho\|_{1,T} \leq \rho_1^*$ and the measurement $\hat{z}(\cdot)$ of $z(\cdot)$ verifies $|\epsilon(t)| \leq \rho_2^*$ for all $t \geq 0$, the following holds.

There exist $\alpha \in \mathcal{K}_\infty$ and $T' > 0$ such that any trajectory $z(\cdot)$ of (3) controlled by $\mathcal{C}(T, \tau, \Sigma)$ with $0 < \tau \leq \tau_0$ and such that $|z(0)| \leq R_0$, satisfies the following:

- (1) $z(t)$ is defined for all $t \geq 0$;
- (2) $|z(t)| \leq \alpha(R_0 + \varepsilon_0) + \varepsilon_0$ for all $t \geq 0$;
- (3) $|z(t)| \leq \varepsilon_0$ for all $t \geq T'$.

Remark 7: We note that Theorem 1 implies that the algorithm $\mathcal{C}(T, \tau, \Sigma)$, with T and τ properly selected, is also robust with respect to perturbations which are small in magnitude (i.e. in the supremum norm). In fact, suppose that T , τ , ρ_1^* and ρ_2^* are as in Theorem 1 and that $z(\cdot)$ is a trajectory in the conditions of that theorem, but that the perturbation $\rho(\cdot)$ is bounded and satisfies $|\rho(t)| \leq \rho^*/T$ for all $t \geq 0$. Then $\rho(\cdot)$ verifies $\|\rho\|_{1,T} \leq \rho^*$, and the conclusions of Theorem 1 hold for that trajectory.

3.2 Switched homogeneous systems

In this section, in addition to the assumptions made so far, we assume that the maps f_i in (1) are homogeneous of degree one. Next we prove the following result, which is the other main one of the paper.

Theorem 2: Suppose that the maps f_i are homogeneous of degree one for all $i = 1, \dots, N$. Suppose also that Σ is a scale-invariant \mathcal{U} -stabilizer of (2). Then there exist $T > 0$ and $\tau_0 > 0$ such that the following holds.

There exist positive constants c_1, c_2 and μ such that, if $z(\cdot)$ is a trajectory of (3) controlled by $\mathcal{C}(T, \tau, \Sigma)$ with $0 < \tau \leq \tau_0$ for which the perturbation $\rho(\cdot)$ in (3) satisfies $\|\rho\|_{1,T} < \infty$, and the measurement error $\epsilon(\cdot)$ is bounded, then

$$|z(t)| \leq c_1 |z(kT)| e^{-\mu(t-kT)} + c_2 \max\{\|\rho_k\|_{1,T}, \|\epsilon_k\|_\infty\} \quad \forall t \geq kT, \forall k \in \mathbb{N}_0, \quad (17)$$

where $\epsilon_k(s) = \epsilon(s + kT)$ and $\rho_k(s) = \rho(s + kT)$ for all $s \geq 0$. In addition

$$\limsup_{t \rightarrow \infty} |z(t)| \leq c_2 \max\{\limsup_{t \rightarrow \infty} |\epsilon(t)|, \lim_{k \rightarrow \infty} \|\rho_k\|_{1,T}\}. \quad (18)$$

Remark 8: Note that due to causality $\|\rho_k\|_{1,T}$ and $\|\epsilon_k\|_\infty$ can be replaced in (17) by, respectively, $\|\rho_k^t\|_{1,T}$ and $\|\epsilon_k^t\|_\infty$, where we define for $t \geq kT$

$$\rho_k^t(s) = \begin{cases} \rho_k(s) & \text{if } 0 \leq s \leq t - kT \\ 0 & \text{if } s > t - kT, \end{cases}$$

and ϵ_k^t in a similar way.

Remark 9: From Remark 6 it follows that the conclusions of Theorem 2 remain valid, if we replace $\|\rho_k\|_{1,T}$ by $\|\rho_k\|_{p,T}$, with $p > 1$, or by $\|\rho_k\|_\infty$, and if we replace c_2 by $\max\{c_2, T^{1/q}c_2\}$ in the case of the norm $\|\cdot\|_{p,T}$ and by $\max\{c_2, Tc_2\}$ in the case of the supremum norm.

In particular, it results that the control system (3) controlled by $\mathcal{C}(T, \tau, \Sigma)$ is exponentially ISS if we see the perturbations ρ and the measurement errors ϵ as inputs. It also results that the closed-loop system is exponentially iISS w.r.t. the perturbation ρ . This assertion follows from the fact that $\|\rho\|_{1,T} \leq \int_0^\infty |\rho(s)| ds$ for all $T > 0$. Nevertheless, the bound obtained in Theorem 2 in terms of the norm $\|\cdot\|_{1,T}$ gives us a better description of the behaviour of the closed-loop system than the bounds given by the ISS or iISS properties. In fact, if we consider the perturbation $\rho(t) = 2n$ if $t \in [n, n + 1/2n]$, $n \in \mathbb{N}$ and 0 elsewhere, we have that $\|\rho\|_{1,1} = 1$, that $\lim_{t \rightarrow \infty} \|\rho_0^t\|_\infty = \infty$ and that $\lim_{t \rightarrow \infty} \int_0^t |\rho(s)| ds = \infty$. From the bound (17) in Theorem 2 with $k = 0$, we have that

$$|z(t)| \leq c_1 |z(0)| e^{-\mu t} + c_2 \max\{1, \|\epsilon_0\|_\infty\} \quad \forall t \geq 0,$$

and hence the states converge to a ball of radius $c_2 \max\{1, \|\epsilon_0\|_\infty\}$. On the other hand, from the ISS and the iISS bounds and by using causality, we obtain

$$|z(t)| \leq c_1 |z(0)| e^{-\mu t} + c_2 \max\{\|\rho_0^t\|_\infty, \|\epsilon_0\|_\infty\} \quad \forall t \geq 0, \quad \text{and}$$

$$|z(t)| \leq c_1 |z(0)| e^{-\mu t} + c_2 \max\left\{\int_0^t |\rho(s)| ds, \|\epsilon_0\|_\infty\right\} \quad \forall t \geq 0,$$

inequalities that do not provide any information about the convergence of the states.

The proof of Theorem 2 requires the following result, whose proof can be obtained *mutatis mutandis* from that of Lemma 2 in (Mancilla-Aguilar and García, 2015).

Lemma 2: Suppose that the maps f_i are homogeneous of degree one for all $i = 1, \dots, N$. Let Σ be a scale-invariant \mathcal{U} -stabilizer of the control system (2). If $z(\cdot)$ is a trajectory of (3) controlled by $\mathcal{C}(T, \tau, \Sigma)$ with measurements $\hat{z}(t)$, and $z(0) \neq 0$, then

- (1) $w(\cdot) = z(\cdot)/|z(0)|$ is a trajectory of (3), with perturbation $\rho(\cdot)/|z(0)|$ instead of $\rho(\cdot)$, controlled by $\mathcal{C}(T, \tau, \Sigma)$ with measurements $\hat{w}(\cdot) = \hat{z}(\cdot)/|z(0)|$.
- (2) For any non-negative integer k , $w(\cdot) = z(\cdot + kT)$ is a trajectory of (3) controlled by $\mathcal{C}(T, \tau, \Sigma)$ with the measurements $\hat{w}(\cdot) = \hat{z}(\cdot + kT)$.

Proof. of Theorem 2. Pick any $\varepsilon_0 \in (0, 1)$, set $R_0 = 1$ and let $T, \tau_0, \rho_1^*, \rho_2^*, T'$ and α be as in Theorem 1. We can assume without loss of generality that $T' = mT$ for some $m \in \mathbb{N}$. Let $\rho_i, i = 1, 2$ be positive real numbers, and let $z(\cdot)$ be a trajectory of (3) controlled by $\mathcal{C}(T, \tau, \Sigma)$ with $0 < \tau \leq \tau_0$ and measurements $\hat{z}(\cdot)$ such that $\|\epsilon\|_\infty \leq \rho_2$. Also suppose that the perturbation $\rho(\cdot)$ in (3) satisfies $\|\rho\|_{1,T} \leq \rho_1$. Finally, let $R = \max_{i=1,2}(\rho_i/\rho_i^*)$.

Suppose that $|z(kT')| \geq R$ for certain $k \in \mathbb{N}_0$. Then the following hold

1. $|z(t)| \leq [\alpha(1 + \varepsilon_0) + \varepsilon_0]|z(kT')|, \forall t \in [kT', (k+1)T']$;
2. $|z((k+1)T')| \leq \varepsilon_0|z(kT')|$.

In order to prove this assertion consider $w(t) = z(t + kT')/|z(kT')|$ for all $t \geq 0$. By Lemma 2, $w(\cdot)$ is a trajectory of (3), with perturbation $\rho(\cdot + kT')/|z(kT')|$ instead of $\rho(\cdot)$, controlled by $\mathcal{C}(T, \tau, \Sigma)$ with measurements $\hat{w}(\cdot) = \hat{z}(\cdot + kT')/|z(kT')|$. From the facts that $|w(0)| = 1$, $|w(t) - \hat{w}(t)| \leq \rho_2/R \leq \rho_2^*$ for all $t \geq 0$ and $\|\rho(\cdot + kT')/|z(kT')|\|_{1,T} \leq \rho_1/R \leq \rho_1^*$, and the selection of T, τ_0 and $\rho_i^*, i = 1, 2$, it follows that

- $|w(t)| \leq \alpha(1 + \varepsilon_0) + \varepsilon_0$, for all $t \in [0, T']$;
- $|w(T')| \leq \varepsilon_0$.

Therefore 1. and 2. hold.

Now, suppose that for some $k \geq 0$, $|z(jT')| \geq R$ for $j = 0, \dots, k$. Then, by applying 1. and 2. above recursively, it follows that

$$|z(t)| \leq [\alpha(1 + \varepsilon_0) + \varepsilon_0]\varepsilon_0^j|z(0)|, \text{ for all } t \in [jT', (j+1)T'], \text{ for all } 0 \leq j \leq k. \quad (19)$$

If we define $c = [\alpha(1 + \varepsilon_0) + \varepsilon_0]/\varepsilon_0$ and $\mu = -\ln(\varepsilon_0)/T'$, (19) implies that

$$|z(t)| \leq c|z(0)|e^{-\mu t}, \quad \forall t \in [0, (k+1)T'].$$

As a consequence, there exists $\bar{k} \in \mathbb{N}_0$ such that

- (i) $|z(t)| \leq c|z(0)|e^{-\mu t}$ for all $t \in [0, \bar{k}T']$;
- (ii) $|z(\bar{k}T')| < R$.

We note that $\bar{k} = 0$ when $|z(0)| < R$.

Next we will show that

$$|z(t)| \leq c(R + \rho_1)e^{LT}, \quad \forall t \geq \bar{k}T', \quad (20)$$

where $L \geq 0$ is a Lipschitz constant for $F(\cdot)$, which exists since F is assumed locally Lipschitz and homogeneous of degree one and therefore globally Lipschitz.

We prove (20) by *reductio ad absurdum*.

Let $J = \{t \geq \bar{k}T' : |z(s)| \leq c(R + \rho_1)e^{LT}, \quad \bar{k}T' \leq s \leq t\}$ and suppose that $\hat{t} = \sup J$ is finite.

Since $|z(\cdot)|$ is continuous, $J = [\bar{k}T', \hat{t}]$. As $|z(\bar{k}T')| < R$, $|z(\hat{t})| = c(R + \rho_1)e^{LT} > R$ (since $c > 1$) and $z(\cdot)$ is continuous, there exists $\tilde{t} \in [\bar{k}T', \hat{t}]$ such that $|z(\tilde{t})| = R$ and $|z(t)| > R$ for all $t \in (\tilde{t}, \hat{t}]$.

Let $k \in \mathbb{N}$ be such that $(\tilde{k} - 1)T \leq \tilde{t} < \tilde{k}T$. From (3) and the fact that $|F(\xi)v| \leq L|\xi|$ for all $\xi \in \mathbb{R}^n$ and all $v \in U$, it follows that

$$|\dot{z}(t)| \leq L|z(t)| + |\rho(t)| \quad \text{a.e. on } [\tilde{t}, \tilde{k}T].$$

Gronwall's Lemma and the fact that $c > 1$ yield for all $t \in [\tilde{t}, \tilde{k}T]$,

$$\begin{aligned} |z(t)| &\leq |z(\tilde{t})|e^{L(t-\tilde{t})} + \int_{\tilde{t}}^t e^{L(t-s)}|\rho(s)| ds \\ &\leq (R + \|\rho\|_{1,T})e^{LT} \leq c(R + \rho_1)e^{LT}. \end{aligned}$$

In consequence, it follows that $\hat{t} > \tilde{k}T$ and that $|z(\tilde{k}T)| \leq c(R + \rho_1)e^{LT}$. Since $|z(t)| \geq R$ for all $t \in [\tilde{k}T, \hat{t}]$, it follows that

$$|z(t)| \leq c(R + \rho_1)e^{LT}e^{-\mu(t-\tilde{k}T)}, \quad \forall t \in [\tilde{k}T, \hat{t}]. \quad (21)$$

In fact, if we consider $w(\cdot) = z(\cdot + \tilde{k}T)$ and $k^* = \lfloor (\hat{t} - \tilde{k}T)/T' \rfloor$, we have that $|w(jT')| \geq R$ for $j = 0, \dots, k^*$. Then, with the same argument employed in the first part of the proof, we prove that

$$|w(t)| \leq c|w(0)|e^{-\mu t}, \quad \forall t \in [0, (k^* + 1)T'],$$

and from the latter and the definitions of $w(\cdot)$ and k^* we arrive to (21).

From (21) it follows that $|z(\hat{t})| < c(R + \rho_1)e^{LT}$. Then, by continuity, there exists $\delta > 0$ such that $|z(t)| < c(R + \rho_1)e^{LT}$ for all $t \in [\hat{t}, \hat{t} + \delta]$, which contradicts the definition of \hat{t} .

Taking into account (i) and (20), we have that

$$|z(t)| \leq \max \{c|z(0)|e^{-\mu t}, c(R + \rho_1)e^{LT}\}, \quad \forall t \geq 0,$$

and then

$$|z(t)| \leq 2c|z(0)|e^{-\mu t} + 2c(R + \rho_1)e^{LT}, \quad \forall t \geq 0.$$

Therefore, by considering the definition of R , it follows that

$$|z(t)| \leq c_1|z(0)|e^{-\mu t} + c_2 \max\{\rho_1, \rho_2\}, \quad \forall t \geq 0, \quad (22)$$

where $c_1 = 2c$ and $c_2 = 2ce^{LT} \frac{1+\rho_1^*}{\min(\rho_1^*, \rho_2^*)}$. Since (22) holds for every $\rho_1 > 0$ such that $\|\rho_1\|_{1,T} \leq \rho_1$ and for every $\rho_2 > 0$ such that $\|\epsilon\|_\infty \leq \rho_2$, we have that

$$|z(t)| \leq c_1|z(0)|e^{-\mu t} + c_2 \max\{\|\rho\|_{1,T}, \|\epsilon\|_\infty\}, \quad \forall t \geq 0. \quad (23)$$

Taking into account (23) and the fact that $z(\cdot + kT)$ is a trajectory of (3), with ρ_k instead of ρ , controlled by $\mathcal{C}(T, \tau, \Sigma)$ with measurements $\hat{z}(\cdot + kT)$, we arrive to (17). Finally, by taking first $\limsup_{t \rightarrow \infty}$ and then $\lim_{k \rightarrow \infty}$ on both sides of (17) we obtain (18) and the thesis holds. \square

The proof of the following result, which considers the case in which the perturbations and measurement/estimation errors are exponentially convergent, can be obtained *mutatis mutandis* from that of Theorem 4 in (Mancilla-Aguilar and García, 2015). Here, the exponential convergence of the perturbations means that $\|\rho(\cdot + t)\|_{1,T} \leq re^{-\alpha t}$ for some positive constants r and α .

Theorem 3: Suppose $\mathcal{C}(T, \tau, \Sigma)$ is a controller for which the conclusions of Theorem 2 hold. Given positive constants k_i and μ_i , $i = 1, 2$, there exist positive constants \tilde{c} and $\tilde{\mu}$ so that any trajectory $z(\cdot)$ of (3) controlled by $\mathcal{C}(T, \tau, \Sigma)$, for which the perturbation $\rho(\cdot)$ in (3) verifies for some $r_1 \geq 0$, $\|\rho(\cdot + t)\|_{1,T} \leq k_1 r_1 e^{-\mu_1 t}$ for all $t \geq 0$, and the measurement error $\epsilon(\cdot)$ verifies for some $r_2 \geq 0$, $\epsilon(t) \leq k_2 r_2 e^{-\mu_2 t}$ for all $t \geq 0$, satisfies

$$|z(t)| \leq \tilde{c} \max(r_1, r_2, |z(0)|) e^{-\tilde{\mu} t} \quad \forall t \geq 0. \quad (24)$$

3.2.1 Observer-based stabilization

In some cases, instead of a measurement we have an estimation $\hat{z}(t)$ of the state $z(t)$ of (3) which is provided by an observer. In the case in which the controller $\mathcal{C}(T, \tau, \Sigma)$ satisfies the conclusions of Theorem 2, it straightforwardly follows that the implementation of the controller by using the estimations $\hat{z}(t)$ given by an observer, practically stabilizes the perturbed system (3), provided the perturbations are bounded in the norm $\|\cdot\|_{1,T}$ and the supremum norm of the estimation error $\epsilon(\cdot) = \hat{z}(\cdot) - z(\cdot)$ is bounded. If, in addition, the estimation error converges to 0 and the perturbation ρ in (3) satisfies $\|\rho(\cdot + t)\|_{1,T} \rightarrow 0$ as $t \rightarrow \infty$, then the states $z(t)$ converge to the origin, and, due to Theorem 3, that convergence is exponential when both the perturbations and the estimation error exponentially converge to zero.

In the case of switched linear systems $\dot{x}(t) = A_{\sigma(t)}x(t)$ with a switched perturbed output $y(t) = C_{\sigma(t)}x + p(t)$ such that every pair (C_i, A_i) is observable, it can be proved, by using arguments similar to those used in the proof of Theorem 5 in (Mancilla-Aguilar and García, 2015), that there exists an observer which provides an estimation $\hat{z}(t)$ of the state $z(t)$ such that $\epsilon(t) = \hat{z}(t) - z(t)$ verifies the following:

- $\epsilon(\cdot)$ is bounded when the system perturbation is bounded in a power norm and the output measurement error $p(\cdot)$ is bounded in the supremum norm;
- $\epsilon(\cdot)$ exponentially converges to zero when both the perturbations and the output measurement errors converge exponentially to zero (see Mancilla-Aguilar and García, 2015, for details).

4. Example

In this section we present a numerical example that shows the behaviour of the proposed controller when it is applied to a switched system with a three-dimensional state-space, and whose subsystems are linear and unstable. The controller is driven by estimations of the state of the switched system, which are obtained from a perturbed output by means of an observer.

Consider the switched system (1) composed by two subsystems $\dot{x} = f_i(x)$, given by $f_1(x) = A_1x$, $f_2(x) = A_2x$ and

$$A_1 = \begin{bmatrix} 0 & 2\pi & 0 \\ -2\pi & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -3 \\ 0 & 3 & -2 \end{bmatrix},$$

with output given by $y(t) = C_{\sigma(t)}x(t) + p(t)$, where

$$C_1 = [1 \ 0 \ 1], \quad C_2 = [1 \ 1 \ 1],$$

and where $p(t)$ is the measurement error. It follows readily that the pairs (C_1, A_1) and (C_2, A_2) are observable.

Let $\{\nu_1, \nu_2, \nu_3\}$ be the canonical base of \mathbb{R}^3 . It is easy to see that the projection of the trajectories $x(t)$ of the first subsystem onto the plane $\text{span}\{\nu_1, \nu_2\}$ are circles and grow exponentially in the direction of the axis ν_3 if $x_3(0) \neq 0$ (see Figure 1). On the other hand, the projection of the trajectories $x(t)$ of the second subsystem onto the plane $\text{span}\{\nu_2, \nu_3\}$ are spirals that converge to the origin of that plane, while the trajectories grow exponentially in the direction of the axis ν_1 if $x_1(0) \neq 0$ (see Figure 2). Let us as in (2) consider the control system associated with these vector fields:

$$\dot{z} = u_1 f_1(z) + u_2 f_2(z) = F(z)u, \quad (25)$$

with $u = [u_1 \ u_2]^T \in \text{co}(\{e_1, e_2\}) \subset \mathbb{R}^2$.

According to the remarks above, given an initial condition $z_0 = (z_{01}, z_{02}, z_{03})$, we consider the following open-loop stabilizing law:

- (1) if $z_{01} = 0$, $u_{z_0}(t) = e_2$ for all $t \geq 0$,
- (2) if $z_{01} \neq 0$, $u_{z_0}(t) = e_1$ for $t \in [0, t_f]$, and $u_{z_0}(t) = e_2$ for all $t \geq t_f$ where t_f is the time elapsed until the trajectory $z(t, z_0, u)$ corresponding to the constant control $u(t) = e_1$ reaches the plane $\text{span}\{\epsilon_2, \epsilon_3\}$.

The family $\Sigma = \{u_{z_0}\}_{z_0 \in \mathbb{R}^3}$ so generated is clearly a scale-invariant \mathcal{U} -stabilizer of system (25). In the simulations below we used this stabilizer in the control algorithm $\mathcal{C}(T, \tau, \Sigma)$.

We assume that the system is perturbed and in consequence, instead of (25) we have

$$\dot{z}(t) = u_1(t)f_1(z(t)) + u_2(t)f_2(z(t)) + \rho(t) = F(z(t))u(t) + \rho(t), \quad (26)$$

with $\rho(t) \in \mathbb{R}^3$ the system perturbation.

We also assume that the estimations of the state $\hat{z}(t)$ of (26) are given by the observer

$$\frac{d\hat{z}}{dt}(t) = u_1(t) [(A_1 + K_1 C_1)\hat{z}(t) - K_1 y(t)] + u_2(t) [(A_2 + K_2 C_2)\hat{z}(t) - K_2 y(t)].$$

Next we present the results of the simulation of the evolution of the switched system when the switching signal $\sigma(t)$ is obtained from the control $u(t)$ given by the tracking controller $\mathcal{C}(T, \tau, \Sigma)$ (see Remark 5). According to the proofs of Theorem 1 (Theorem 11 in (Mancilla-Aguilar and García, 2013)) and Theorem 2, the values of $T \approx 20.2$ sec. and $\tau \approx 10^{-10}$ sec. should be used to guarantee the stabilization. Nevertheless, by a trial and error process we found that the algorithm works properly with considerably larger values of τ and smaller of T , showing the conservativeness of the bounds (of worst case type) obtained in that theorems. Some of the parameters taken in the simulation are: initial condition of the system $z_0 = (6, 4, 8)$, initial condition of the observer $\hat{z}_0 = (3, 2, 4)$ tracking period $T = 1$ sec., dwell-time $\tau = 0.01$ sec. and time of simulation $T_{sim} = 20$ sec. The gains of the observer were taken as

$$K_1 = \begin{bmatrix} -10.812 \\ -7.215 \\ -5.188 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -11.667 \\ -3.167 \\ 2.833 \end{bmatrix}.$$

In this way the poles of $(A_1 + K_1 C_1)$ and $(A_2 + K_2 C_2)$ are -4 , -5 and -6 . In the simulations we assumed that the measurement error is $p(t) = r(t)$ with $r(t)$ a random process with uniform distribution on $(-5/2, 5/2)$. The euclidean norm $|\rho(t)|$ of the system perturbation is shown in Figure 3, where we note that $\|\rho\|_{1,T} = 1$, while $\|\rho\|_\infty = 25$.

Figure 4 shows the components of the state $z(t)$ of the switched system, while Figure 5 shows

the euclidean norm of the state $|z(t)|$ and of the observer $|\hat{z}(t)|$. In this last figure it can be clearly seen the exponential decay of the states and that they remain bounded but do not converge to zero. Finally in Figure 6 we present the stabilizing switching signal $\sigma(t)$ and in Figure 7 a detail of this signal can be seen.

5. Conclusions

We have shown that the algorithm for the stabilization of switched systems introduced in (Mancilla-Aguilar and García, 2013) stabilizes switched systems (exponentially in the case of homogeneous switched systems) in a robust way with respect to perturbations which are bounded, with an integral bound. These results comprise, among others, both the exponential ISS and the exponential iISS properties of the closed-loop system. Nevertheless, they are stronger than those stability properties, since a better description of the behavior of the closed-loop system is obtained, in the case in which the perturbations are unbounded and non-integrable on $[0, +\infty)$.

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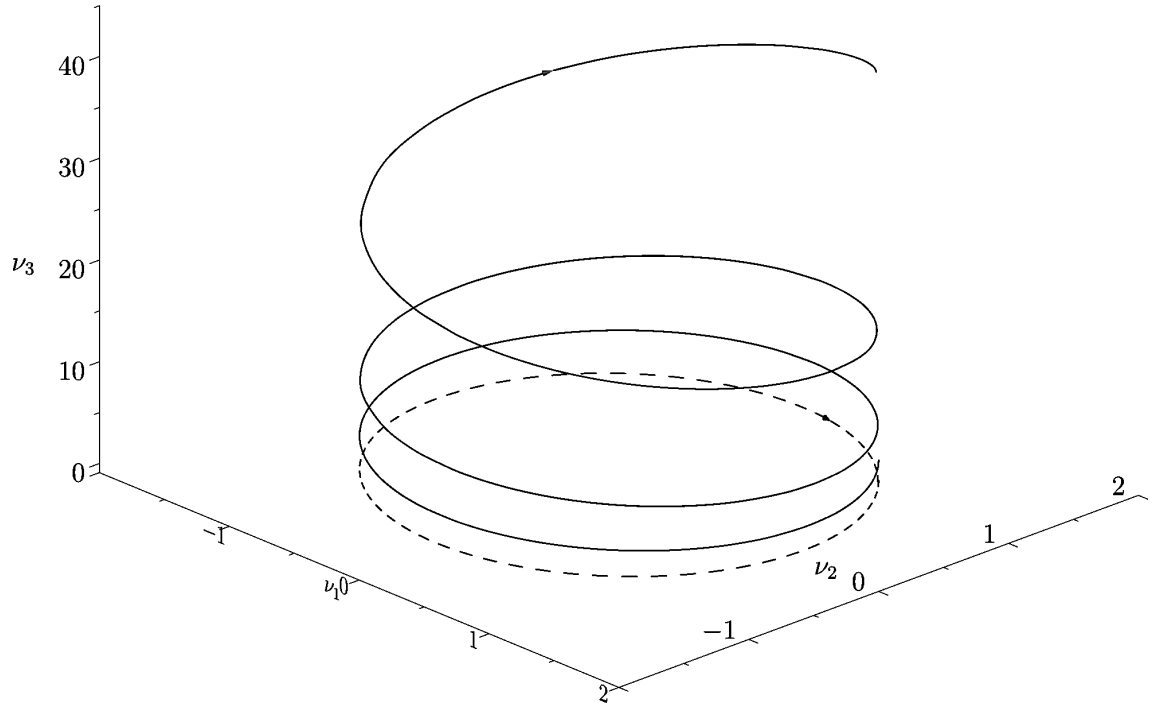


Figure 1. Trajectory of the first subsystem (solid) and its projection onto the plane $\text{span}\{\nu_1, \nu_2\}$ (dashed)

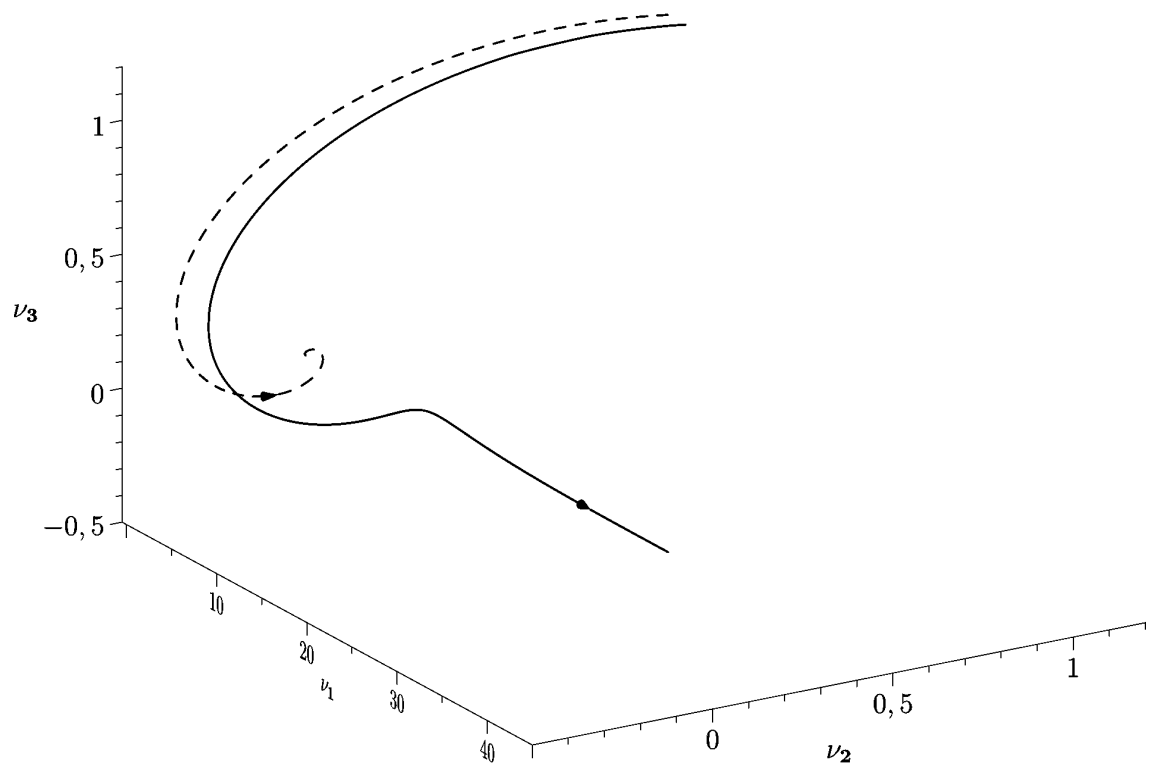


Figure 2. Trajectory of the second subsystem (solid) and its projection onto the plane $\text{span}\{\nu_2, \nu_3\}$ (dashed)

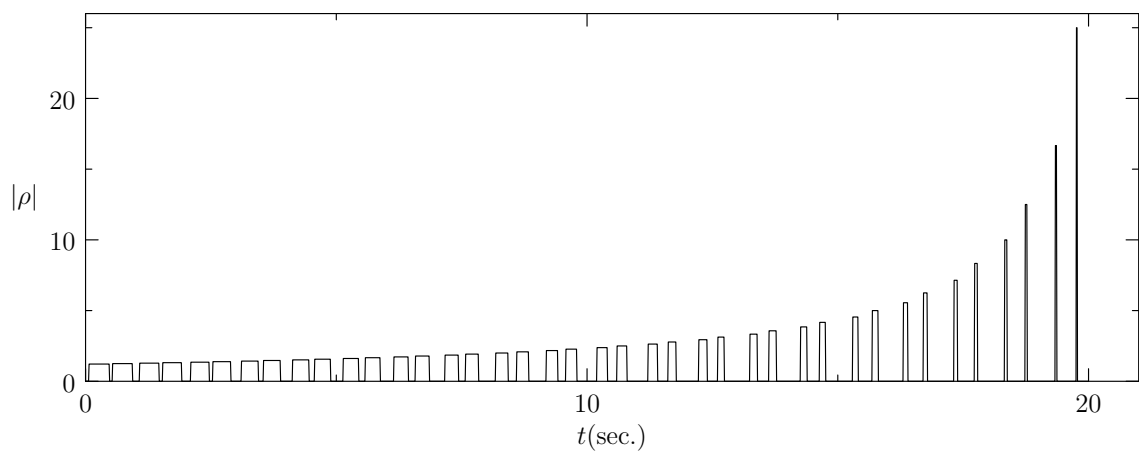


Figure 3. Model perturbation $|\rho(t)|$

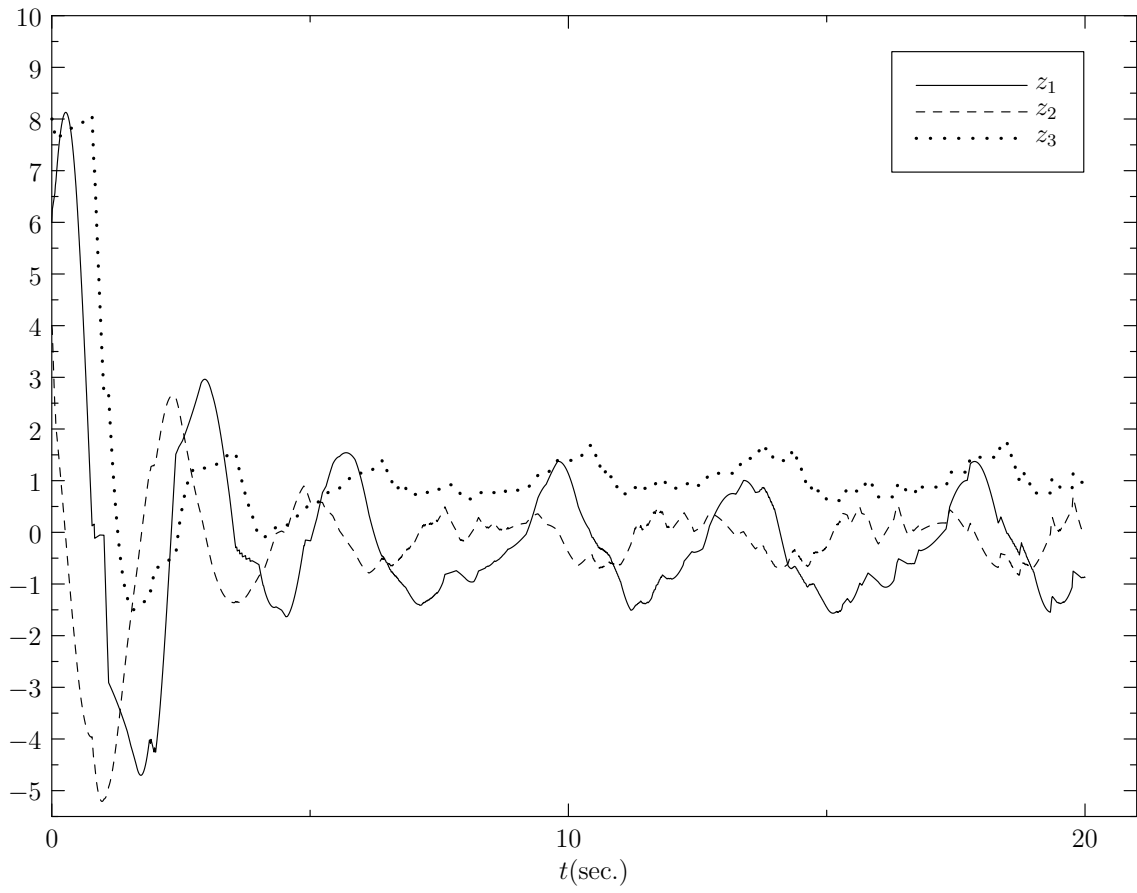


Figure 4. State components of $z(t)$.

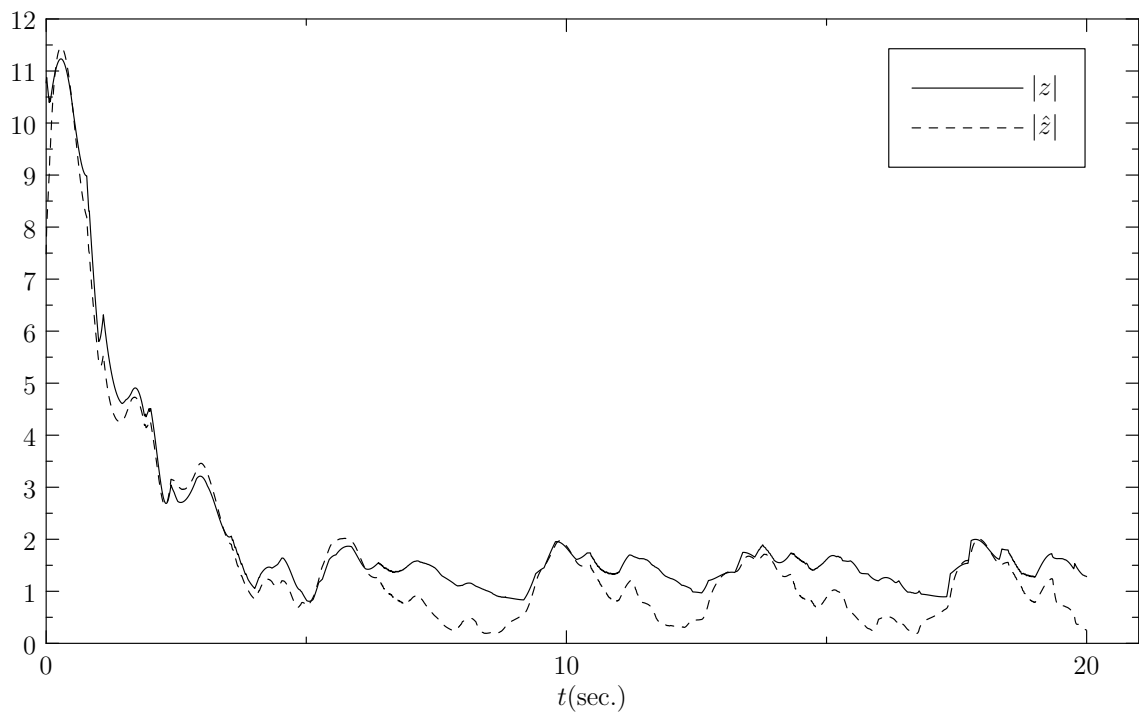


Figure 5. State and observer norms $|z(t)|$ and $|\hat{z}(t)|$.

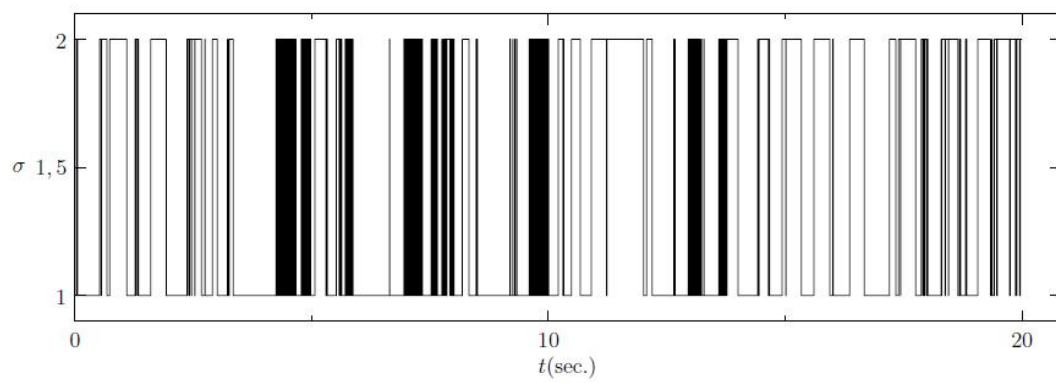


Figure 6. Switching signal

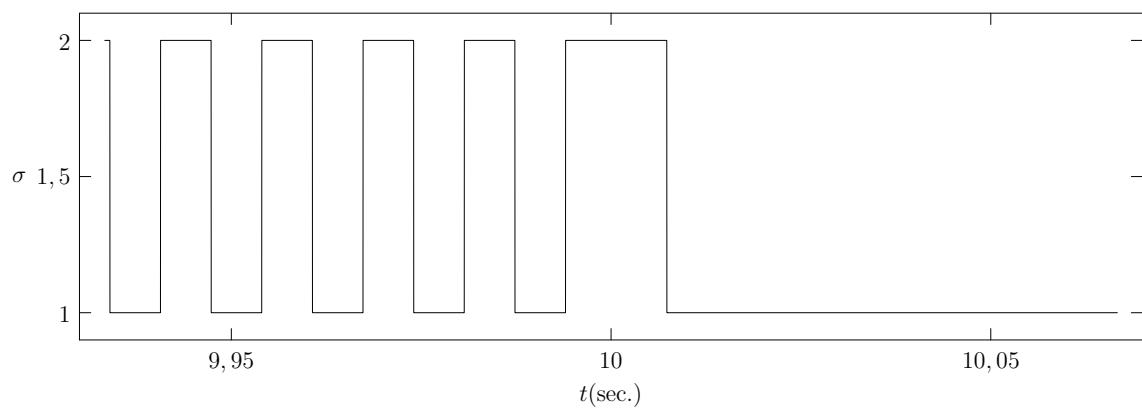


Figure 7. Switching signal (detail)