Analysis and Design of a Tilted Rotor Hexacopter for Fault Tolerance

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A proof is presented of how a hexagon-shaped hexacopter can be designed to keep the ability to reject disturbance torques in all directions while counteracting the effect of a failure in any of its motors. The method proposed is simpler than previous solutions, because it does not require change of the motor rotation direction or in-flight mechanical reconfiguration of the vehicle. It consists of tilting the rotor a small fixed angle with respect to the vertical axis. Design guidelines are presented to calculate the tilt angle to achieve fault-tolerant attitude control without losing significant vertical thrust. It is also formally proved that the minimum number of unidirectional rotating motors needed to have fault tolerance is 6 and that this can be achieved by tilting their rotors. This proof is essentially a control allocation analysis that recovers in a simple way a result already known: the standard configuration (without tilting the motors) is not fault tolerant. A simulation example illustrates the theory.

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I. INTRODUCTION

In recent years, multirotor microaerial vehicles (MAVs) have become popular because the electronic systems needed to fly them have dramatically increased their availability and usefulness, decreasing their cost and weight. Simplicity and cost-effectiveness have turned out to be appealing, and as a consequence, an increasing number of applications have risen in many fields, such as agriculture, surveillance, and photography. As mission requirements become more demanding, the matter of fault tolerance should be taken into account, especially if controller certification is sought.

A recent survey on fault-tolerant techniques for unmanned aerial vehicles (UAVs), in particular for multirotor vehicles, can be found in [1]. Several failure detection techniques and fault-tolerant controls (FTCs) have been applied to develop fault-tolerant multirotor vehicles. In [2], an $\mathcal{H}_\infty$ loop-shaping control technique is used to stabilize a quadrotor in case of rotor failure. This is accomplished by letting the yaw axis be uncontrolled. FTC for vehicles with different numbers of rotors has been investigated. In [3], the FTC problem has been studied for multirotor vehicles with 4, 6, and 8 rotors. In [4], an error detection and fault isolation technique based upon nonlinear observers is presented for the 8-rotor helicopter case. A linear parameter varying (LPV) approach has been proposed to control an 8-rotor helicopter in case of rotor failures [5]. In [6], the LPV technique is also used to solve the FTC problem for a quadrotor. A thorough literature review on FTC for UAVs can be found there as well. Model predictive control has also been suggested to deal with failures in multirotor vehicles [7, 8]. Control allocation has been studied in [9] and [10], and sliding mode-based control has been used in [11–13].

Within the literature on MAV fault tolerance, different definitions can be found. Some references, such as [14], treat the matter of fault tolerance with respect to position control. The minimum number of rotors needed to achieve different kinds of fault tolerance is an issue that has been also discussed in [15] and [16], among others. It is clear that at minimum, 4 rotors are needed to fly a multirotor MAV with the ability to reject disturbance torques in any direction. Although it is possible to fly with fewer than 4 rotors [14], it can be shown that in that case, torques cannot be exerted in certain directions.

Vehicles with more than 4 rotors frequently can be found. In [16], a study on the trade-offs among number of rotors, maneuverability, efficiency, and redundancy is carried out. The analysis covers three MAVs of different shapes and configuration. As explained there, vehicles

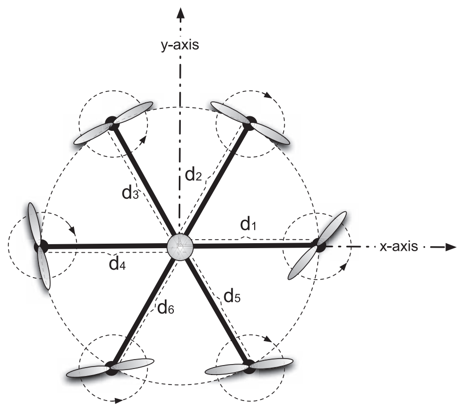

Fig. 1. Hexacopter axes in standard configuration. Top view.

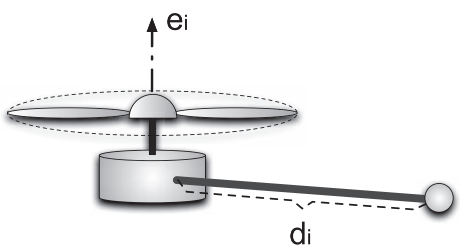

Fig. 2. Hexacopter arm detail in standard configuration.

with 6 or 8 rotors could turn out to be more reliable, achieving safer performance in general and eventually fault tolerance. Fault tolerance is analyzed for the three cases, including the hexagon-shaped, 6-rotor helicopter, the kind of vehicle treated here.

In case of failure in one rotor, a hexagon-shaped hexacopter with unidirectional rotors, such as the one shown in Figs. 1 and 2, will see its performance degraded, because the attitude controller will be unable to reject disturbance torques in certain directions [14–16]. In [17], a so-called degraded control strategy is analyzed in case of rotor failure, the approach consisting of leaving the yaw axis uncontrolled. This approach is also suggested in [16]. But this may not be enough for certain applications.

In [16], a design approach rendering fault tolerance is presented for the hexagon-shaped hexacopter. The key to this solution is the use of bidirectional rotors. As shown there, in case of motor failure, it is not possible with five unidirectional motors to generate torques in pitch or roll without generating a residual disturbing torque in the yaw axis. Yaw control can be achieved only with the ability to invert the spinning direction of the rotor opposite to the one that is faulty. The approach is general to a certain extent, because the motors' electronic speed controllers (ESCs) must be capable of stopping and starting the motor in the opposite direction. This kind of ESC is not usual in commercially available MAVs at present.

Here, the possibility of having bidirectional rotors is ruled out. Apart from it being uncommon to find bidirectional ESCs, it is expected that changes in the spinning direction of any rotor could introduce undesirable disturbances. As explained later, to overcome the lack of fault tolerance of the hexagon-shaped hexacopter with unidirectional rotors, a symmetric tilt of the axes of rotation of all rotors is suggested. This allows for fault tolerance with no need to resort to bidirectional ESCs.

The approach followed here to study the fault tolerance problem is worthy of clarification as well. While fault tolerance in multirotor MAVs has been previously approached as a dynamic system's controllability problem (see, for instance, [17]), here it is approached as an actuator allocation problem. Although the discussion based upon controllability criteria is richer, the study of the actuator allocation problem is more general, because it is not dependent on the applied control technique. Looking into the actuator allocation problem is a step that should be taken before getting into issues concerning dynamics and control.

Within the actuator allocation framework, this research is based upon [18], where the analysis and design of the minimum number of thrusters needed to achieve complete attitude control under thruster failure in spacecraft is presented. The results presented in [18] are applicable to spacecraft that use thrust actuators. Here, the main difference with the previous paper is that while a thruster is only capable of exerting a force in a given direction, a motor is capable of exerting a force and a torque at the same time. This changes the problem significantly.

In the present article, as in [18], the fault tolerance criterion followed leans toward the single failure point criterion [19]. According to this, in case of failure in one rotor, the vehicle should be able to continue operating with no significant change in its capability to fulfill its mission. The vehicle will be required to maintain control in the 4 usually controlled degrees of freedom (DOFs), namely, altitude, roll, pitch, and yaw, in the case of a single failure. As also shown in [16], it is formally proved that the standard (vertical) rotor arrangement for hexagon-shaped hexacopters cannot withstand a failure in any of its rotors without losing its ability to control 1 of the 4 failure-controlled DOFs, hence degrading performance. With the standard rotor arrangement, 1 DOF is lost in the vehicle attitude after a failure. In addition, here it is proved that by tilting all motors with respect to the vertical alignment toward the vehicle's z-axis (Figs. 1 and 2), FTC can be achieved without losing control in any of the aforementioned 4 DOFs.

While seeking to hold attitude and altitude control (4 DOFs) in case of failure, rotor tilt renders another effect worth mentioning when not under failure. While nonfaulty hexagon-shaped hexacopters without rotor tilt provide control in no more than the usual 4 DOFs, the tilted rotor configuration is theoretically capable of providing 6-DOF control when there are no motor failures. Even though this is worthy of comment, it is out of the scope of this research to investigate FTC in anything but attitude and altitude. The adopted approach responds to many of today's vehicles and applications being based upon vehicles that provide 4-DOF attitude and altitude control. Moreover, it is clear that this 6-DOF control capability of the hexagon-shaped helicopter with 6 unidirectional tilted rotors is lost immediately in case of rotor failure.

In addition to aforementioned issues, a distinction should be made between the qualitative and the quantitative aspects of the analysis. A qualitative result presented here is that in case of failure in one motor, by tilting the motors, disturbance torques up to a certain magnitude can be rejected in any direction. However, the maximum magnitude of the worst-case torque is computed here, which turns out to be a quantitative result. The quantitative analysis introduced in the present article leads to a methodology to design the tilt angle of the hexacopter's motors, taking into consideration the worst-case disturbance torque and the minimum vertical thrust necessary to keep the hexacopter airborne. A control allocation scheme based upon the Moore-Penrose pseudoinverse is used to determine the relation between vertical thrust and worst-case torque perturbation during normal operation and in case of motor failure. This is used to design the most convenient tilt angle.

The paper is organized as follows. In Section II, the necessary background and notation that are used throughout the paper are presented. Section III presents the main results concerning the standard and the tilted rotor configurations. The proofs are moved to the appendix to simplify the reading flow. In Section IV, a simulated example is presented that illustrates the usefulness of the results. Concluding remarks are drawn in Section V.

II. NOTATION AND BACKGROUND

A. Notation

$\mathbb{R}_+$ represents positive real numbers. Given $X \in \mathbb{R}^{n\times m}$, $N(X)$ denotes its kernel, $R(X)$ its range, X^T its transpose, and $X^\dagger$ its Moore-Penrose pseudoinverse. The ijth component of matrix $X \in \mathbb{R}^{n\times m}$ is denoted x_{ij}, the ith row is represented by $x_i^r \in \mathbb{R}^m$, and the jth column is shown as $x_j^c \in \mathbb{R}^n$. The ith element of vector $v \in \mathbb{R}^m$ is denoted v_i. Given $m \in \mathbb{N}$ vectors $v_k^c \in \mathbb{R}^n, k = 1, \ldots, m$, $X = [v_i^c]_{i=1,m}$ is the matrix $X \in \mathbb{R}^{n\times m}$ with columns v_i^c. Given $X \in \mathbb{R}^{n\times m}$, $\tilde{X}_j \in \mathbb{R}^{n\times(m-1)}$ represents the matrix with its jth column eliminated.

A vector $u \in \mathbb{R}^n$ is called positive (strictly positive), and denoted $u \geq 0$ ($u > 0$), if its components are positive, i.e., $u_1 \geq 0, \ldots, u_n \geq 0\,(u_1 > 0, \ldots, u_n > 0)$. The Euclidean norm of a vector $u \in \mathbb{R}^n$ is denoted as $\|u\| = (\sum_{i=1}^n u_i^2)^{1/2}$, and its maximum norm is denoted by $\|u\|_\infty = \max_{1\leq k\leq n}|u_k|$. The notations $c\theta = \cos(\theta)$ and $s\theta = \sin(\theta)$ are used to make some equations look more compact.

Throughout the paper, a failure in the second motor is considered. This holds for failures in any other motor due to symmetry considerations.

B. Background

Previous results that analyze the minimum number of thrusters necessary to achieve complete attitude FTC with thrusters [18] are presented for clarity. Suppose that we have $n \in \mathbb{N}$ thrusters capable of generating forces arranged in vector $p \in \mathbb{R}^n$ such that $p \geq 0$ and that $A \in \mathbb{R}^{3\times n}$ is a matrix relating these forces with the torque acting on the vehicle. More specifically, if $q \in \mathbb{R}^3$ is the torque produced by forces p, then

$$q = Ap. \tag{1}$$

Matrix A depends on thruster location and orientation with respect to the center of mass of the vehicle.

THEOREM 1 *(See Theorem 1 in [18])* The following conditions are equivalent.

1) For each $q \in \mathbb{R}^3$ there exists $p \geq 0$ such that $q = A_p$.
2) Matrix A has full rank, and there exists $w \in N(A)$ with strictly positive components, i.e., $w > 0$.

If conditions of Theorem 1 are fulfilled, we say that the n thrusters solve the torque and force problem.

C. Vehicle Model

Each rotor on the vehicle exerts a force and a torque, which are given for $i = 1, \ldots, n$ as follows:

$$\begin{aligned} p_i &= k_f u_i \\ r_i &= k_t u_i \\ u &= \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}^T \end{aligned}$$

Here, u_i is the pulse width modulation (PWM) percentage commanded to the ith motor. The force and torque linear dependence with the PWM percentage is often a reasonable assumption [15]. Constants k_f and k_t relate the PWM command to the norm p_i of each force and to the norm r_i of each torque of each rotor. As usual, it is assumed that all motors are identical. Next, we state the total vehicle force and torque in the body frame coordinates (Figs. 1 and 2):

$$f = k_f E u, \qquad q = \left(k_t E J + k_f H\right) u$$
$$E = \left[e_i^c\right]_{i=1,n}, \quad H = \left[d_i^c \times e_i^c\right]_{i=1,n} \triangleq \left[h_i^c\right]_{i=1,n}$$

Here, the location of the center of mass of the ith motor is given by $d_i^c \in \mathbb{R}^3$, and the direction of the corresponding force is given by $e_i^c \in \mathbb{R}^3$. Both vectors are represented in body frame coordinates. J is a diagonal matrix with diagonal entries $j_{ii} = (-1)^{i+1}$, for $i = 1, \ldots, n$, indicating that Rotors 1, 3, and 5 spin counterclockwise (CCW) while the remaining Rotors 2, 4, and 6 spin clockwise (CW). The CW and CCW rotor spin directions are defined from an observer's perspective, which looks at the vehicle from above (Fig. 1):

$$A = k_t \begin{bmatrix} -c\theta & \frac{1}{2}\left(c\theta + \frac{\sqrt{3}\ell k_f s\theta}{k_t}\right) & \frac{1}{2}\left(c\theta + \frac{\sqrt{3}\ell k_f s\theta}{k_t}\right) & -c\theta & \frac{1}{2}\left(c\theta - \frac{\sqrt{3}\ell k_f s\theta}{k_t}\right) & \frac{1}{2}\left(c\theta - \frac{\sqrt{3}\ell k_f s\theta}{k_t}\right) \\ -\ell\frac{k_f}{k_t}s\theta & \frac{1}{2}\left(\sqrt{3}c\theta - \frac{\ell k_f s\theta}{k_t}\right) & \frac{1}{2}\left(\frac{\ell k_f s\theta}{k_t} - \sqrt{3}c\theta\right) & \ell\frac{k_f}{k_t}s\theta & \frac{1}{2}\left(\sqrt{3}c\theta + \frac{\ell k_f s\theta}{k_t}\right) & \frac{1}{2}\left(-\sqrt{3}c\theta - \frac{\ell k_f s\theta}{k_t}\right) \\ s\theta & -s\theta & s\theta & -s\theta & s\theta & -s\theta \end{bmatrix} \tag{2}$$

III. MAIN RESULTS

A. Standard Configuration

In Theorem 1, the necessary and sufficient condition for a vehicle to solve the torque and force problem is based upon the existence of a positive vector in the kernel of the A matrix in (1). In Theorem 2 of [18], this result is applied to the single-failure situation in any of the vehicle's actuators. For the multirotor helicopter, this theorem can be rewritten as follows.

THEOREM 2 The torque and force problem can be solved with a failure in rotor j if and only if $\tilde{A}_j \in \mathbb{R}^{n\times m-1}$ has full rank and there exists $w > 0$ such that $\tilde{A}_j w = 0$.

As a result of this theorem, [18] proves that the minimum number of actuators that allow the torque and force problem to be solved with tolerance to failure in one actuator is $n = 6$. In the case of multirotor helicopters, it seems natural to consider $n = 6$ the minimum number of rotors needed to deal with a single rotor failure. However, it is well known that the standard hexacopter configuration, i.e., with forces exerted upward and with unidirectional rotating motors, is not completely fault tolerant [16, 17]. This means that it does not solve the torque and force problem after a rotor failure. Next, a proof of this fact is presented that differs from the previous arguments presented in [16] and [17]. The idea behind this proof allows the introduction of a simple modification in the standard hexacopter configuration to achieve a complete fault tolerance.

Even though the results presented in [18] can be applied to study a hexacopter's fault tolerance, it should be taken into account that additional constraints appear because matrix A in (1) has a certain structure. So, the first question that should be answered is the following: Is there a matrix $A = k_t EJ + k_f H$ satisfying the conditions in Theorem 2 for every $i = 1, \ldots, 6$? This is a general result that may lead to different rotor configurations such that the conditions of the theorem are satisfied. Here, for practical reasons, the approach focuses on the typical symmetric configuration, i.e., with the rotors distributed in a hexagon. The analysis begins with the classical configuration NPNPNP, i.e., odd-numbered motors rotating CW (N) and even-numbered motors rotating CCW (P). Here, all rotors are identical, and their thrust and torque are exerted in the direction of the vehicle's z-axis (Figs. 1 and 2). In this case, we have

$$d_i^c = \ell \begin{bmatrix} c\alpha_i \\ s\alpha_i \\ 0 \end{bmatrix}, \quad e_i^c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad h_i^c = \begin{bmatrix} \pm\ell s\alpha_i \\ \mp\ell c\alpha_i \\ 0 \end{bmatrix}$$

where $\alpha_i = (i-1)\frac{\pi}{3}$ rad; $i = 1, \ldots, 6$; and $\ell > 0$ is the distance to each vertex of the hexagon. As a consequence,

$$A = \begin{bmatrix} 0 & \frac{k_f\ell\sqrt{3}}{2} & \frac{k_f\ell\sqrt{3}}{2} & 0 & -\frac{k_f\ell\sqrt{3}}{2} & -\frac{k_f\ell\sqrt{3}}{2} \\ -k_f\ell & -\frac{k_f\ell}{2} & \frac{k_f\ell}{2} & k_f\ell & \frac{k_f\ell}{2} & -\frac{k_f\ell}{2} \\ k_t & -k_t & k_t & -k_t & k_t & -k_t \end{bmatrix}.$$

Here, A has full rank and

$$N(A) = \operatorname{span}\left\{ \begin{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T, \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}^T, \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}^T \end{matrix} \right\}.$$

By Theorem 1, this configuration solves the torque and force problem: $w[1\ 1\ 1\ 1\ 1\ 1]^T \in N(A)$. However,

$$N\left(\tilde{A}_2\right) = \operatorname{span}\left\{ \begin{matrix} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \end{bmatrix}^T, \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \end{bmatrix}^T \end{matrix} \right\}.$$

Thus, if $\tilde{w} \in N(\tilde{A}_2)$, then $\tilde{w}_4 = 0$, and by Theorem 2, it follows that this configuration cannot solve the torque and force problem with a failure in any of its rotors. Recall that $\tilde{A}_2$ equals the A matrix with its second column removed. The $\tilde{w}_4$ component corresponds to the fifth rotor of the vehicle, the one that is opposite the faulty one.

EXAMPLE 1 Suppose that $\ell = k_f = k_t = 1$, and let $q = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & -1 \end{bmatrix}^T$. The vectors $u \in \mathbb{R}^6$ satisfying $Au = q$ are given by

$$u = A^\dagger q + s,$$

where $s \in N(A)$. Observe that

$$A^\dagger q = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 \end{bmatrix}^T.$$

Because u must be positive, the (minimum norm) solution is given by

$$\begin{aligned} u_0 &= A^\dagger q + \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}^T \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T \geq 0. \end{aligned}$$

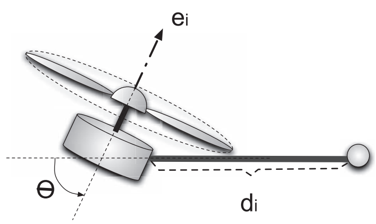

Fig. 3. Tilted motor configuration.

But if rotor $i = 2$ fails, because

$$\tilde{u}_0 = (\tilde{A}_2)^{\dagger} q = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

and $\tilde{w}_4 = 0$ for every $\tilde{w} \in N(\tilde{A}_2)$, there is no $u \geq 0$ such that $\tilde{A}_2 u = q$. Thus, the control command cannot be executed.

The preceding results do not depend on the selected configuration (NPNPNP), because if we make a permutation of the third row in the matrix, the kernel of the resulting matrix is the same. Thus, for all configurations in which the motors spin with respect to the vehicle's z-axis, a hexacopter cannot fully control attitude with a failure in one of its rotors. The next section is devoted to the study of hexacopters with tilted rotors.

B. Tilted Configuration

Because we assume that the probability of failure is the same for each motor, it is reasonable to propose a symmetric configuration (Figs. 1 and 2). When considering a tilt of the rotors on the vehicle, a θ angle is defined as the angle between the d_i^c and the e_i^c vectors with the tilt being the same for all rotors ($i = 1, \ldots, 6$). In the standard configuration, with all e_i^c vectors aligned with the z-axis of the body frame, the angle is $\theta = \pi/2$. In this section, it is instead assumed that each rotor's axis, given by the e_i^c vector, is tilted a (fixed) angle $(\pi/2 - \theta)$ toward the corresponding d_i^c arm, with $0 < |\pi/2 - \theta| < \pi/2$ (Fig. 3). As a consequence, $A = A(\theta)$ is given by (2).

It is not hard to see that A has full rank and $w[1\ 1\ 1\ 1\ 1\ 1]^T \in N(A)$ for any $0 \leq |\theta - \pi/2| < \pi/2$. So, the torque and force problem can be solved in the nominal case, i.e., with no failures. In addition, with this configuration, the torque and force problem can be solved after a failure in any rotor. Because of the symmetry, it is enough to prove this statement for any rotor, for instance, $i = 2$. First, we prove that $\tilde{A}_2$ has full rank,

$$\det\left(\tilde{A}_2 \tilde{A}_2^T\right) = \frac{27}{4} k_t^2 (s\theta)^2 \left[c\,(2\theta)\left(k_t^2 - \ell^2 k_f^2\right) + \ell^2 k_f^2 + k_t^2\right]^2 \neq 0, \quad (3)$$

for every $0 \leq |\theta - \pi/2| < \pi/2$.

Now, because a failure in rotor $i = 2$ was assumed, a vector $w = [\,w_1\ w_2\ w_3\ w_4\ w_5\,]^T > 0$, $w \in N(\tilde{A}_2)$ should be found. It can be seen that such a vector $w \in N(\tilde{A}_2)$ satisfies the following:

$$w_1 = \frac{1-\alpha}{\alpha+1} - w_4 \frac{1-\alpha}{2\alpha} \quad (4)$$

$$w_2 = 1$$

$$w_3 = 1 - \frac{1-\alpha}{2\alpha} w_4$$

$$w_5 = w_4 + \frac{1-\alpha}{\alpha+1}, \quad (5)$$

where $\alpha = \frac{k_t}{\sqrt{3}\ell k_f \tan\theta}$ is defined. Here, $\alpha \neq -1$, $\alpha \neq 0$ has been assumed and, without loss of generality, $w_2 = 1$. Observe that $|\alpha| \to 0$, i.e., $\theta \to \pi/2$, is a desired condition, because it maximizes thrust. Although it is possible to find $w > 0$ for every $0 < |\theta - \pi/2| < \pi/2$, for practical reasons (to maximize the thrust in the z-axis) and to avoid unnecessary calculations, it has been assumed that θ is such that $|\alpha| < 1$. We analyze this assertion in the next lemma.

LEMMA 1 Suppose that $0 < |\alpha| < 1$, i.e., $|\tan\theta| > \frac{k_t}{\sqrt{3}\ell k_f}$, and let $w \in \mathbb{R}^5$ as defined in (4) to (5). Then, if $0 < w_4 < |2\alpha/(1+\alpha)|$, it follows that $w > 0$.

PROOF See the appendix.

The preceding lemma shows that there exists a (strictly) positive vector in $N(\tilde{A}_2)$ if $0 < |\alpha| < 1$. As mentioned previously, an analysis of the case in which $|\alpha| > 1$ is avoided because $|\alpha|$ is as small as possible is sought. Nevertheless, if $|\alpha| > 1$, there is also a (strictly) positive vector in $N(\tilde{A}_2)$. However, if $|\alpha| = 1$, it is not possible to find such a vector. It can be shown that if $\alpha = 1$, then $w_1 = 0$ (something similar happens when $\alpha = -1$). Next, the main result of this section is presented.

LEMMA 2 For a symmetric and tilted configuration of motors with angle $\theta \neq \pi/2$ and $|\tan\theta| \neq \frac{k_t}{\sqrt{3}\ell k_f}$, the hexacopter can reject perturbation torques in any direction in $\mathbb{R}^3$ to maintain its attitude, even with the failure of one of its rotors.

PROOF It is a consequence of Lemma 1, (3), and Theorem 2.

This is an interesting result from the qualitative point of view, because it shows that tilting the rotors allows full control of the attitude of the multirotor vehicle with one faulty rotor, without major variations (change the spinning direction of its rotors or a mechanical reconfiguration). Furthermore, a hexarotor has the minimum number of rotors to achieve this property. However, as explained in Section I, a quantitative analysis also needs to be made to compute a practical value for the tilt angle θ; hence, the vehicle's thrust should be considered.

C. Thrust Equations

Let $v > 0$ be the thrust of the hexacopter. Thrust v depends on the PWM signals $u \geq 0$ in the following way:

$$v = k_f s\theta \mathbf{1}^T u \quad \text{with} \quad \mathbf{1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T.$$

The mapping $u \to (q, v)$ is given by

$$\begin{bmatrix} q \\ v \end{bmatrix} = B(\theta)u = \begin{bmatrix} A(\theta) \\ k_f s\theta \mathbf{1}^T \end{bmatrix} u\,. \tag{6}$$

With the standard configuration ($\theta = \pi/2$), the vertical thrust is maximized. On the one hand, as shown previously, fault tolerance for the torque and force problem cannot be achieved in this case. On the other hand, for every $0 < |\pi/2 - \theta| < \pi/2$ and $|\tan\theta| \neq \frac{k_t}{\sqrt{3}\ell k_f}$, it is possible to solve the torque and force with tolerance to one faulty rotor. A question that naturally arises at this point concerns the criterion for determining the θ angle. It is expected to have a trade-off in the selection of θ between the capability to reject torque disturbances and the ability to exert vertical thrust on the vehicle.

To address this issue, suppose Rotor 2 is faulty, as before. The usual approach for the allocation of torque control commands q, if it exists, is to compute the actuator signal u of minimum norm. In case of failure in one rotor, there will be among all possible torque commands a torque in a given direction that will demand an actuator signal u of maximum norm. This particular worst-case torque command q_{wc}, whose direction induces a maximum over all minimum norm u actuator signals, will depend on the θ angle. In the case of $\theta = \pi/2$, as the torque gets closer to the worst-case direction, the norm of vector u needed to allocate such a torque tends to infinity.

Suppose that based upon practical considerations, a given bound $q_{\max} > 0$ is set on the torque commands whose allocation is sought. Within all torques $q \in \mathbb{R}^3$ with $\|q\| < q_{\max}$, the following θ-dependent function is proposed:

$$f(\theta) = \max_{\substack{q \in \mathbb{R}^3 \\ \|q\| < q_{\max}}} \min_{\substack{\tilde{B}_j u = q \\ u \geq 0}} \|u\|$$

where $\tilde{B}_j = \tilde{B}_j(\theta)$ corresponds to matrix $B(\theta)$ with a failure in the jth rotor, as defined in Section I.

For typical multirotors, the torques about the x and y directions are more important than those about the z direction. This is because angular accelerations about x and y change the vehicle's thrust direction and therefore jeopardize position control [15, 17]. Thus, a weighted norm could be considered for $\|q\|$ to prioritize the x and y directions.

The objective is to compute the curve $f(\theta)$, such as the one indicated in Fig. 4 that plots the minimal motor forces $\|u\|$ needed to reject the worst-case perturbation torques under motor failure. The worst-case torque q_{wc} for $\theta = \pi/2$ can be obtained analytically; instead, q_{wc} for a variable angle has been obtained numerically.

The general idea is to determine a practical way to design the motor slant angle based on the worst-case perturbation torque to be rejected and the minimum vertical thrust that maintains the hexacopter flying. From Fig. 4, it can be observed that as θ approaches $\pi/2$, the minimal force $u \geq 0$ needed to reach the worst-case torque

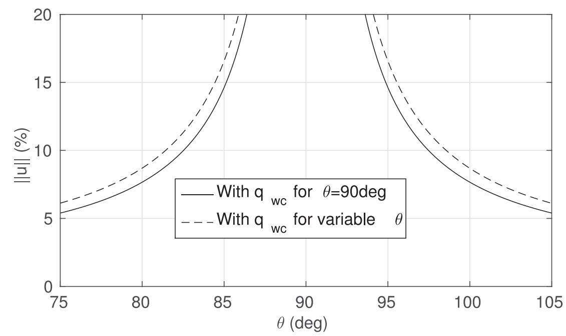

Fig. 4. Minimum force for worst-case torques.

rapidly increases. However, as θ moves above or below $\pi/2$, the thrust is reduced according to $1/s\theta$. This establishes a compromise between the thrust reduction that can be afforded by tilting the rotors and the maximum perturbation torque that can be rejected after a failure of one rotor.

Although previous results provide criteria to design the geometry of the vehicle, $u \geq 0$ does not consider the vehicle vertical thrust, and this should be taken into account. The minimum norm of $u \geq 0$ to solve the torque and force problem is computed, but in practice, this force $u \geq 0$ is chosen in such a way that it guarantees certain torque $q \in \mathbb{R}^3$ and vertical thrust $v > 0$.

D. Actuator Allocation

1) *Nominal Case*: Assuming no rotor failures, to allocate a given torque and thrust pair $(q, v) \in \mathbb{R}^3 \times \mathbb{R}_+$, the actuators' signal $u \in \mathbb{R}^6$ is usually chosen as follows:

$$u_0 = B(\theta)^\dagger \begin{bmatrix} q \\ v \end{bmatrix} = \begin{bmatrix} A(\theta) \\ k_f s\theta \mathbf{1}^T \end{bmatrix}^\dagger \begin{bmatrix} q \\ v \end{bmatrix} \tag{7}$$

The reason to allocate u using the Moore-Penrose pseudoinverse is that it renders the minimum norm solution. Other solutions based on generalized pseudoinverses can improve the control allocation at the expense of a higher real-time computational cost [20].

Although it is possible to prove that for a given pair (q, v) there exists a positive solution $u \geq 0$ of (6) because of the existence of positive vectors in $N(A(\theta))$, the positiveness of u_0 is not guaranteed. Let $C = A^\dagger$, and observe that because $A = A(\theta)$ is full rank,

$$B(\theta)^\dagger = \begin{bmatrix} A \\ k_f s\theta \mathbf{1}^T \end{bmatrix}^T \left(\begin{bmatrix} A \\ k_f s\theta \mathbf{1}^T \end{bmatrix} \begin{bmatrix} A^T & k_f s\theta \mathbf{1} \end{bmatrix} \right)^{-1}.$$

Because $\mathbf{1} \in N(A)$, it follows that

$$u_0 = \begin{bmatrix} A^T (AA^T)^{-1} & \frac{1}{6k_f s\theta}\mathbf{1} \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix} = \begin{bmatrix} C & \frac{1}{6k_f s\theta}\mathbf{1} \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix}.$$

Then, a necessary and sufficient condition for the positiveness of u_0 is

$$\left| \min_{i=1,\ldots,6} \left(c_i^r\right)^T q \right| \leq \frac{v}{6k_f s\theta}. \tag{8}$$

A consequence of (8) is that for each torque q, it gives a lower bound on the total thrust $v = v(\theta)$ to guarantee $u_0 \geq 0$. A more restrictive condition that simplifies the calculations would consist of a thrust that does not depend on q, i.e., a thrust that assures $u_0 \geq 0$ in the worst case.

REMARK 1 Because a bound for the thrust is given in terms of $C = A(\theta)^{\dagger}$, the following equation is useful for $0 < |\theta - \pi/2| < \pi/2$:

$$C = \begin{bmatrix} -\frac{k_t c\theta}{3\left(k_t^2(c\theta)^2+\ell^2k_f^2(s\theta)^2\right)} & -\frac{\ell k_f s\theta}{3\left(k_t^2(c\theta)^2+\ell^2k_f^2(s\theta)^2\right)} & \frac{1}{s\theta 6k_t} \\ \frac{k_t c\theta+\sqrt{3}\ell k_f s\theta}{6k_t^2(c\theta)^2+6\ell^2k_f^2(s\theta)^2} & \frac{\sqrt{3}k_t c\theta-\ell k_f s\theta}{6k_t^2(c\theta)^2+6\ell^2k_f^2(s\theta)^2} & -\frac{1}{s\theta 6k_t} \\ \frac{k_t c\theta+\sqrt{3}\ell k_f s\theta}{6k_t^2(c\theta)^2+6\ell^2k_f^2(s\theta)^2} & \frac{\ell k_f s\theta-\sqrt{3}k_t c\theta}{6k_t^2(c\theta)^2+6\ell^2k_f^2(s\theta)^2} & \frac{1}{s\theta 6k_t} \\ -\frac{k_t c\theta}{3\left(k_t^2(c\theta)^2+\ell^2k_f^2(s\theta)^2\right)} & \frac{\ell k_f s\theta}{3k_t^2(c\theta)^2+3\ell^2k_f^2(s\theta)^2} & -\frac{1}{s\theta 6k_t} \\ \frac{k_t c\theta-\sqrt{3}\ell k_f s\theta}{6k_t^2(c\theta)^2+6\ell^2k_f^2(s\theta)^2} & \frac{s\theta\left(\sqrt{3}k_t c\theta+\ell k_f s\theta\right)}{6k_t^2(c\theta)^2+6\ell^2k_f^2(s\theta)^2} & \frac{1}{s\theta 6k_t} \\ \frac{k_t c\theta-\sqrt{3}\ell k_f s\theta}{6k_t^2(c\theta)^2+6\ell^2k_f^2(s\theta)^2} & -\frac{\sqrt{3}k_t c\theta+\ell k_f s\theta}{6\left(k_t^2(c\theta)^2+\ell^2k_f^2s^2(\theta)\right)} & -\frac{1}{s\theta 6k_t} \end{bmatrix}$$

Because of the symmetry of the vehicle structure, every row of C has the same norm. For every $i = 1, \ldots, 6$,

$$\|c_i^r\| = \frac{1}{6\, s\theta}\sqrt{\frac{(s\theta)^2\left(\ell^2k_f^2 + 4k_t^2\right) + k_t^2(c\theta)^2}{\ell^2k_f^2k_t^2(s\theta)^2 + k_t^4(c\theta)^2}}.$$

LEMMA 3 Suppose that $0 < |\theta - \pi/2| < \pi/2$, and let $q_{\max} > 0$. Given a torque and thrust pair (q, v), let u_0 as given in (7). Then, $v > 0$ guarantees $u_0 \geq 0$ for every torque $\|q\| \leq q_{\max}$ if and only if

$$v \geq k_f\, q_{\max}\sqrt{\frac{(s\theta)^2\left(\ell^2k_f^2 + 4k_t^2\right) + k_t^2(c\theta)^2}{\ell^2k_f^2k_t^2(s\theta)^2 + k_t^4(c\theta)^2}}.$$

PROOF See the appendix.

2) *Failure Case*: When rotor j fails, it is not as simple to find a condition as the one in (8) because $\mathbf{1} \notin N(\tilde{A}_j(\theta))$. However, with additional calculations, we can find similar conditions.

Consider first the configuration with the rotors not tilted ($\theta = \pi/2$). As shown in Example 1, if

$$q = \frac{q_{\max}}{\sqrt{2}}\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & -1 \end{bmatrix}^T,$$

there is no $u \geq 0$ such that $\tilde{A}_j(\pi/2)u = q$. Hence, if rotor j fails, there cannot be a bound similar to (8). Now suppose that $0 < |\theta - \pi/2| < \pi/2$, and define the following:

$$\tilde{B}_j(\theta) = \begin{bmatrix} \tilde{A}_j(\theta) \\ k_f s\theta \mathbf{1}^T \end{bmatrix}, \qquad \tilde{B}_j^{\dagger}(\theta) = \begin{bmatrix} M & N \end{bmatrix}$$

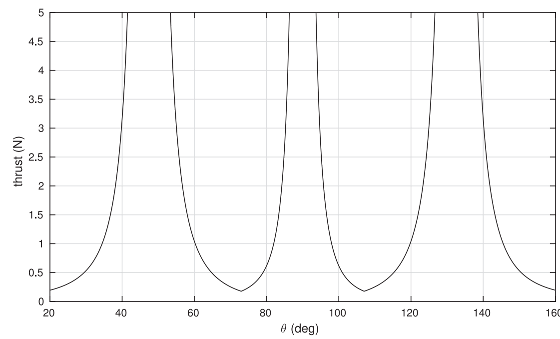

Fig. 5. Minimum thrust for worst-case torques.

for any $j = 1, \ldots, 6$, $M \in \mathbb{R}^{5\times 3}$, $N \in \mathbb{R}^5$, and $\mathbf{1} \in \mathbb{R}^5$. Given the torque and thrust pair (q, v), let

$$u_0 = \tilde{B}_j^{\dagger}(\theta)\begin{bmatrix} q \\ v \end{bmatrix}. \tag{9}$$

As a consequence, $u_0 \geq 0$ if and only if

$$M\, q + N\, v \geq 0.$$

As in the case without faulty rotors, a lower bound on $v > 0$ is sought such that the existence of $u_0 \geq 0$ can be guaranteed for every $\|q\| < q_{\max}$ with $q_{\max} > 0$. The bound can be obtained if the inequality

$$n_i\, v \geq q_{\max}\, \|m_i^r\| \tag{10}$$

is satisfied for every $i = 1, \ldots, 5$, where m_i^r is the ith row of M and n_i is the ith element of vector N. This result is formalized in the next lemma.

LEMMA 4 Suppose that $0 < |\theta - \pi/2| < \pi/2$ and $|\tan\theta| \neq \frac{k_t}{\sqrt{3}\ell k_f}$. Given a torque and thrust pair (q, v), let $u_0 \in \mathbb{R}^5$ as given in (9). Then, there exists $v > 0$ such that $u_0 \geq 0$ for every torque $\|q\| \leq q_{\max}$. Moreover, $v > 0$ guarantees $u_0 \geq 0$ for every torque $\|q\| \leq q_{\max}$ if and only if

$$v \geq q_{\max} \max_{i=1,\ldots,5} \frac{\|m_i^r\|}{n_i}. \tag{11}$$

Equation (11) provides a practical design tool to determine the tilt angle θ based on minimum vertical thrust v and maximum perturbation torque $q_{\max}$.

REMARK 2 In Lemma 3, the existence of $v > 0$ is evident; however, in this case, it is not, because we must prove that $n_i > 0$ for every $i = 1, \ldots, 5$; otherwise, (10) does not hold unless $q_{\max} = 0$. Thus, $n_i > 0$ if and only if $0 < |\theta - \pi/2| < \pi/2$ and $|\tan\theta| \neq \frac{k_t}{\sqrt{3}\ell k_f}$.

PROOF See the appendix.

EXAMPLE 2 Fig. 5 shows, for different values of θ, the minimum thrust $v > 0$ that guarantees $u_0 \geq 0$ for the worst-case torque, with $q_{\max} = 0.01$ Nm, $k_f = 0.0667\frac{\mathrm{N}}{\%}$, $k_t = 0.0346\frac{\mathrm{Nm}}{\%}$, and $d = 0.275$ m. It can be seen that for $\theta \to \pi/2$ and $|\tan\theta| \to \frac{k_t}{\sqrt{3}\ell k_f}$, the thrust tends to infinity.

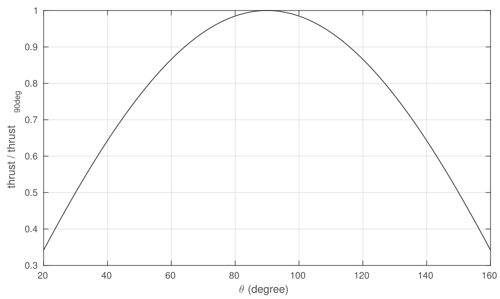

Fig. 6. Loss of thrust (with respect to thrust pointing in z direction (thrust$_{90\text{deg}}$)) as function of θ after failure of one rotor.

TABLE I
Vehicle's Parameters

Quantity	Symbol	Value
Roll/pitch inertia	$I_{xx/yy}$	0.06 kgm^2
Yaw inertia	I_{zz}	0.12 kgm^2
Mass	m	1.4 kg
Arm length	ℓ	0.275 m
PWM/force constant	k_f	$0.0667 \frac{\text{N}}{\%}$
PWM/torque constant	k_t	$0.0346 \frac{\text{Nm}}{\%}$

In this case, the minimum of the required thrust is for $\theta_m = 1.27$ rad $= 73°$. But, the loss of thrust if the vehicle is configured with $\theta = 73°$ must be analyzed. In Fig. 6, it can be seen that for $\theta = 73°$, a loss of 5% of thrust is expected.

The preceding example shows that by losing 5% of thrust, a perturbation in any direction can be rejected with a failure in one rotor. In the next section, it is shown through simulations that it is possible to reject a torque disturbance in the case of a hexacopter with its rotors tilted so that $\theta = 73°$.

IV. EXAMPLE

To test the control allocation results proposed in this work, simulations were carried out. The simulated vehicle is based upon a DJI F550 hexacopter. Parameters such as the vehicle's inertia matrix, its mass, the arm length ℓ, and the k_f and k_t motor constants were either obtained from lab experiments or given from practical assumptions. Their values are shown in Table I, with the inertia matrix given by

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}.$$

Simulations were run considering Rotor 2 is not operational. When configured with the rotors tilted, the θ angle was set to 73° in the simulator.

To carry out simulations that properly show the vehicle's behavior in a realistic setup, a position control loop is engaged with the attitude controller in a way similar to the one proposed in [21] and [22]. Recall that with the rotors tilted, the vehicle is no longer an underactuated system as long as no failure takes place. On the one hand, (6) is the mapping from actuator signals to torque and thrust. On the other hand, the actuator configuration with the rotors tilted is capable of exerting lateral forces (in the body frame) as follows:

$$\begin{bmatrix} f_x \\ f_y \end{bmatrix} = -c(\theta) M\, u, \tag{12}$$

with f_x and f_y being the lateral forces along the vehicle's x and y axes and with M being defined as follows:

$$M = \begin{bmatrix} 1 & c(\pi/3) & -c(\pi/3) & -1 & -c(\pi/3) & c(\pi/3) \\ 0 & s(\pi/3) & s(\pi/3) & 0 & -s(\pi/3) & -s(\pi/3) \end{bmatrix}.$$

As a consequence, under no motor failure and in the case of rotor tilt, a 6×6 bijective mapping takes place between the 6 actuator signals and the 3 torques plus 3 forces. Because vector $\mathbf{1} \in \mathbb{R}^{\mathbf{6}}$ is in the kernel of matrix $B(\theta)$ and M, a positive solution can be found, rendering a fully actuated system. Looking into this fully actuated characteristic of the system with rotor tilt under no motor failures is out of the scope of this research, where fault tolerance in torque and thrust is sought. Most frequently, without rotor tilt, position control of multirotor helicopters must be approached as an underactuated control problem. This is the approach adopted here: the example shows fault tolerance in torque and thrust, with the attitude and altitude control paired with a position controller that follows the most frequent underactuated approach.

The simplified position dynamics are given by

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = -g \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{Z}{m} \begin{bmatrix} c\psi\ s\vartheta\ c\varphi + s\psi\ s\varphi \\ s\psi\ s\vartheta\ c\varphi - c\psi\ s\varphi \\ c\vartheta\ c\varphi \end{bmatrix}, \tag{13}$$

where φ, ϑ, and ψ are the roll, pitch, and yaw angles with respect to a local north-west-up (NWU) frame [21]. Z is the vehicle's total thrust, and $g = 9.8\,\text{m/sec}^2$ is the gravity constant. For clarity, and without losing generality, further suppose the vehicle's desired heading is $\psi = 0°$ (heading north). Moreover assume the roll and pitch angles are small (near hovering control). Let $Z_0 = g \cdot m$ be the vehicle's nominal thrust so that a simpler version of the position dynamics can be drawn from (13) as follows:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} \frac{Z_0}{m}\vartheta \\ -\frac{Z_0}{m}\varphi \\ -g + \frac{Z_0+\delta Z}{m} \end{bmatrix} = \begin{bmatrix} \frac{Z_0}{m}\vartheta \\ -\frac{Z_0}{m}\varphi \\ \frac{\delta Z}{m} \end{bmatrix},$$

with $\delta Z = Z - Z_0$ being the differential thrust. The lateral forces of (12) exerted on the vehicle, either with or without motor failure, are not considered in this model to keep the solution for position control simple. Nevertheless, the simulation takes full account of these forces.

A discussion on different approaches for position control is out of the scope of this work. The reason for including position control here is to show that the control allocation scheme is fault tolerant as far as attitude and altitude control are concerned. Treating the position problem through the underactuated approach allows for

TABLE II
PID Controllers' Constants

Loop	P	I	D
x/*y* position	$K_{xy}^{p} = 0.04$	$K_{xy}^{i} = 0.01$	$K_{xy}^{d} = 0.09$
Altitude	$K_{z}^{p} = 1.00$	$K_{z}^{i} = 0.50$	$K_{z}^{d} = 8.00$
Roll/pitch	$K_{\varphi\vartheta}^{p} = 2.00$	$K_{\varphi\vartheta}^{i} = 3.00$	$K_{\varphi\vartheta}^{d} = 1.00$
Yaw	$K_{\psi}^{p} = 12.00$	$K_{\psi}^{i} = 6.00$	$K_{\psi}^{d} = 3.00$

position control under failure as well, as shown in this example.

A consequence of the adopted solution is an architecture consisting of a cascaded control scheme (see [23] and [24], Chap. 10), with the roll and pitch controllers as inner (or secondary) loops and the *x*/*y*-position controllers as outer (or primary) loops. The altitude and heading loops are designed to be considered independent controls.

With this scheme in mind, a loop at a time *x*/*y*-position control can be designed, with the commanded pitch angle ϑ being the control action signal for the x (forward) direction and the commanded roll angle φ being the control action signal for the y (lateral) direction. To carry out the simulations, proportional–integral–derivative (PID) control was designed for this task so that the commanded pitch and roll angles ϑ_c and φ_c are given as

$$\vartheta_c = K_{xy}^{d}\dot{e}_x + K_{xy}^{p}e_x + K_{xy}^{i}\int_0^t e_x(\nu)d\nu$$

$$\varphi_c = -\left(K_{xy}^{d}\dot{e}_y + K_{xy}^{p}e_y + K_{xy}^{i}\int_0^t e_y(\nu)d\nu\right),$$

where $e_x = x_d - x$ and $e_y = y_d - y$ are the forward and lateral position errors, respectively, and x_d and y_d are the desired forward and lateral position set points, respectively. For altitude control, another PID controller is employed, with the Z thrust given as

$$Z = Z_0 + \delta Z = Z_0 + K_z^d\dot{e}_z + K_z^p e_z + K_z^i\int_0^t e_z(\nu)d\nu,$$

where $e_z = z_d - z$ is the altitude error. On each of the roll, pitch, and yaw axes, three other PID control laws are implemented so that

$$q_\varphi = K_{\varphi\vartheta}^{d}\dot{e}_\varphi + K_{\varphi\vartheta}^{p}e_\varphi + K_{\varphi\vartheta}^{i}\int_0^t e_\varphi(\nu)d\nu$$

$$q_\vartheta = K_{\varphi\vartheta}^{d}\dot{e}_\vartheta + K_{\varphi\vartheta}^{p}e_\vartheta + K_{\varphi\vartheta}^{i}\int_0^t e_\vartheta(\nu)d\nu$$

$$q_\psi = K_{\psi}^{d}\dot{e}_\psi + K_{\psi}^{p}e_\psi + K_{\psi}^{i}\int_0^t e_\psi(\nu)d\nu$$

where q_φ, q_ϑ, and q_ψ are the PID computed torque commands and $e_\varphi = \varphi_c - \varphi$, $e_\vartheta = \vartheta_c - \vartheta$, and $e_\psi = \psi_c - \psi$ are the attitude error angles corresponding to roll, pitch, and yaw, respectively. The values of the constants of all PID controllers can be seen in Table II.

Throughout the simulation, the objective is to maintain the $x = 0$, $y = 0$ horizontal position heading north ($\psi = 0^\circ$) at $z = 40$ m of altitude. Pitch and roll turn out to be commanded by the *x*/*y*-position controllers. The simulations begin with the vehicle at $z = 40$ m of altitude, while the rest of the 6-DOF state variables begin at the 0 value.

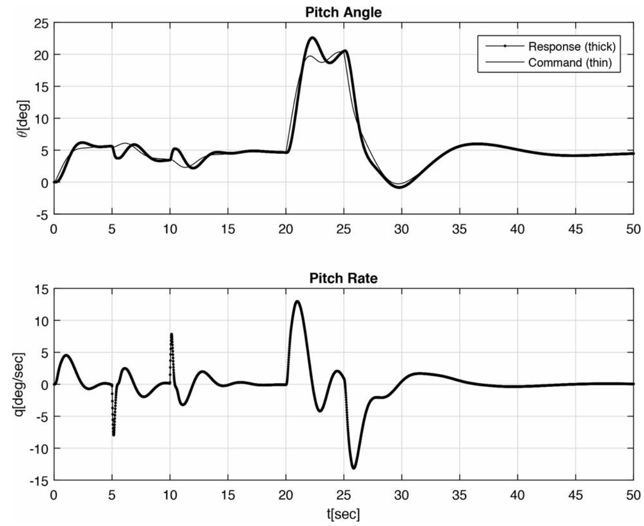

Fig. 7. Simulated ideal case without sensors' noise: Vehicle's pitch angle and rate.

To stimulate and show the simulated transient behavior of the vehicle, a 5-s pulse shaped disturbance torque of the value -0.1 Nm is exerted on the pitch axis (see the upper-right curve later in Fig. 9). Later during the simulation, a 5-s pulse shaped disturbance force of the value 3 N is exerted on vehicle from behind along the vehicle's x-axis (see the lower-right curve later in Fig. 9). Without losing generality, both disturbances cause transient behavior on the vehicle's pitch dynamics.

As far as the vehicle's instrumentation is concerned, two cases are shown. First, the ideal case in which the vehicle's state variables are available for feedback is presented (Figs. 7 and 9). Second, a more realistic simulation setup with noisy measurements is shown (Figs. 11 and 13). In the latter case, the vehicle's attitude is determined from a simulated noisy inertial measurement unit and a simulated noisy magnetometer through a complementary filter (see [25] and references therein). The attitude sensors were simulated with band-limited white noise with standard deviations: 0.4 m/s^2 for the accelerometer, 0.01 rad/s for the gyros, and 0.2° for the magnetometer. Position is supposed to be measured with 5-m rms error noise in the x and y directions, and altitude is supposed to be measured with 2-m rms error noise. Complementary filters are frequently found in practice for attitude estimation in this kind of application. This kind of strategy for attitude estimation relies on assuming that rotor drag effects can be neglected [25].

Without losing generality with respect to the main results presented in this work, the simulations carried out here do not consider rotor dynamics or other aerodynamic effects. As far as the results of the simulations are concerned, the following comments are due.

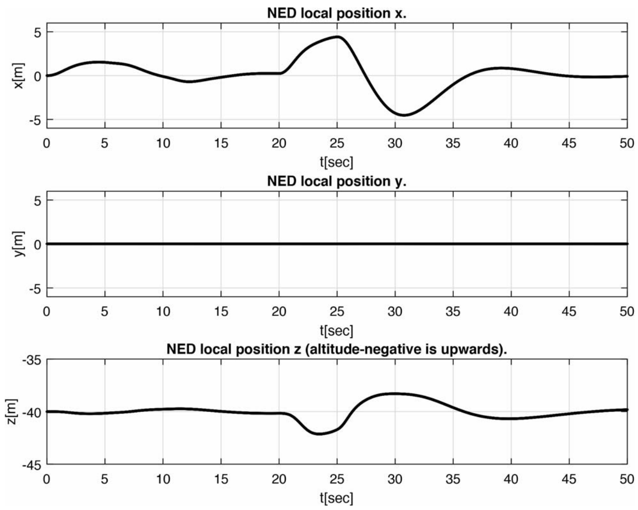

Fig. 8. Simulated ideal case without sensors' noise: Vehicle's position with respect to north-east-down (NED) local frame; desired (x, y) position is (0, 0) m, and desired altitude is 40 m.

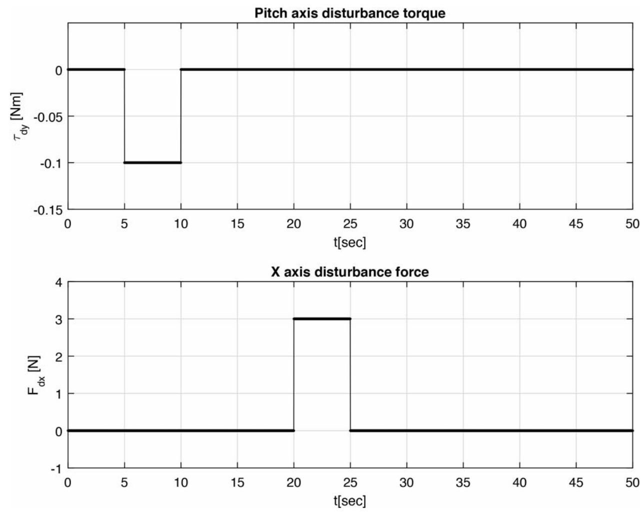

Fig. 10. Pulse disturbance torque and force applied with and without considering sensors' noise.

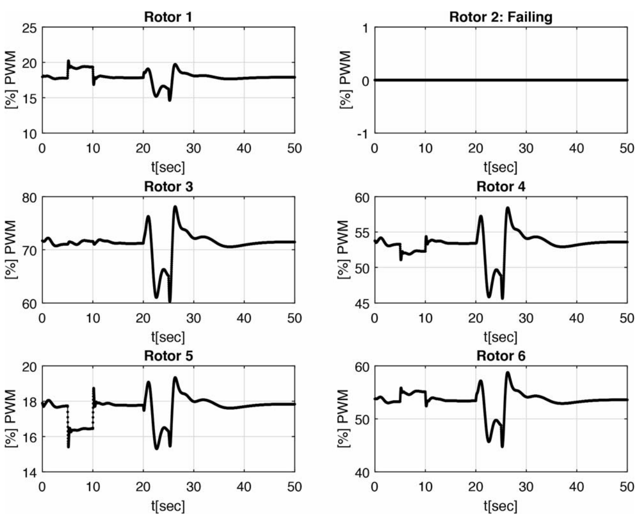

Fig. 9. Simulated ideal case without sensors' noise: Vehicle's PWM motor control signals.

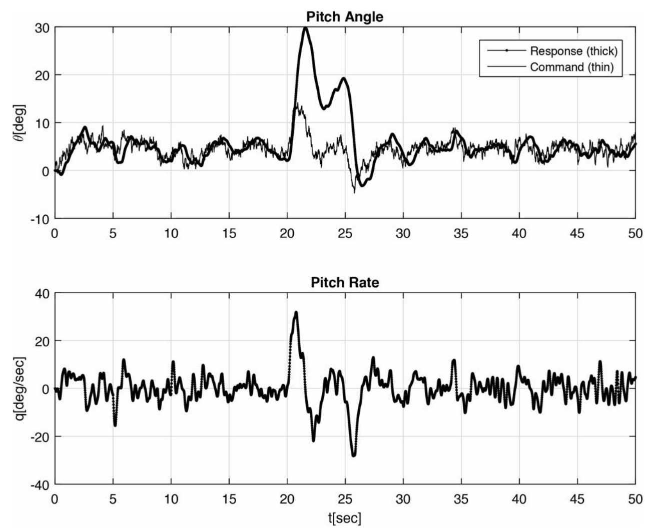

Fig. 11. Simulations with sensors' noise: Vehicle's pitch angle and rate.

Ideal Instrumentation In Fig. 7, the pitch and pitch rate responses of the vehicle can be seen. Together with the pitch response, the pitch angle commanded by the x-position controller can be seen (thin line), which shows the inner controller adequately tracks attitude commands. Integral action makes the x-position controller command a pitch angle of approximately 5°, which is needed to compensate for the lateral force that shows up because of the failure in Rotor 2. In Fig. 8, it can be seen that the position controller properly fulfills its goal, as does the altitude controller.

In Fig. 9, the PWM percentage signals can be seen for each motor, being 0% for Rotor 2. Actuator signal values are within reasonable bounds. In Fig. 10, the disturbances are shown.

Realistic Instrumentation The results of the simulation carried out considering a realistic instrumentation setup show no big differences with respect

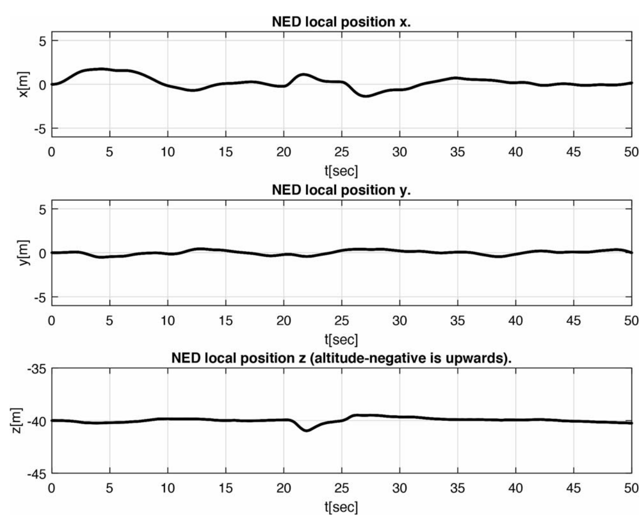

Fig. 12. Simulations with sensors' noise: Vehicle's position with respect to NED local frame; desired (x, y) position is (0, 0) m, and desired altitude is 40 m.

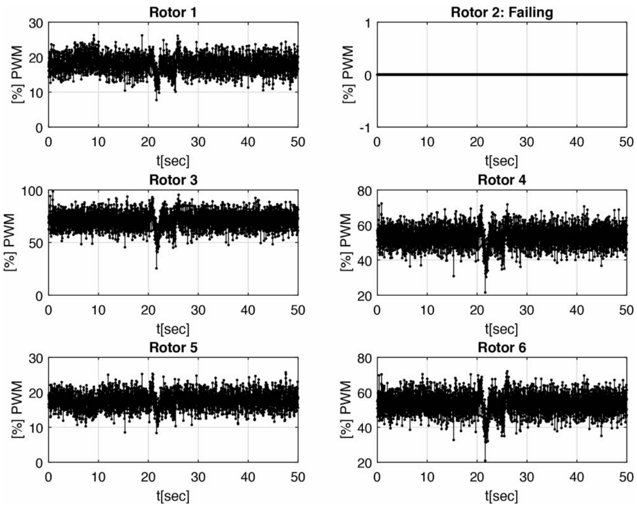

Fig. 13. Simulations with sensors' noise: Vehicle's PWM motor control signals.

to the ideal case. The attitude responses become a little noisier (Fig. 11), while the position responses are practically the same (Fig. 12). The attitude commanded by the position controller (thin trace, upper curves of Fig. 11) shows a certain discrepancy between the command and the response when the lateral force disturbance shows up, because the vehicle's attitude is estimated based upon a complementary filter from accelerometers and gyros data, where lateral forces are not accounted for. On the simulated operation under failure, the lateral force that shows up because of rotor failure is compensated in the complementary filter through feedforward. Actuator PWM percentage signals shown in Fig. 13 show no major differences with respect to the ideal case, other than a noisier response that is nevertheless within reasonable bounds.

V. CONCLUSIONS

The importance of fault tolerance in flight control can never be overemphasized. As more sophisticated applications appear, with more demanding requirements, the need for fault-tolerant systems becomes more evident. In the case of a hexagon-shaped hexacopter with unidirectional rotors, the standard configuration, with the rotors exerting thrust vertically, has been analyzed, with a formal proof of the lack of fault tolerance. A new formal proof has been presented, which shows that a simple modification, consisting of tilting all rotors by a small angle toward the MAV's vertical axis, is enough to achieve fault tolerance. Little vertical thrust is lost—and hence little battery time—while significant improvement is gained, from the qualitative point of view, as the hexacopter becomes truly fault tolerant.

An argument questioning this solution could ask whether tilting the rotors is an approach with any generality. Even though a hexagon-shaped hexacopter with tilted rotors is uncommon, commercially available platforms already introduce some tilt on the arms of the vehicle with respect to the center plates. It is quite clear that lowering the vehicle's center of mass and improving its camera's field of view are the goals of this tilting in platforms such as the DJI S900 [26]. If tilting the hexacopter's arms renders fault tolerance, it is doubly beneficial.

A thorough analysis of the results in this work could lead to design guidelines regarding what kind of configuration is more convenient as far as the choice of motor, propeller, or ESC sets is concerned. This should be paired with a sound choice of arm length and rotor tilt angle so that the loss of vertical thrust because of fault-tolerant design is minimized.

Finally, in this work, it has been shown that when looking into fault-tolerant design, regardless of the discussion on the matter of controllability, actuator allocation is an issue that should be taken care of first.

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APPENDIX. PROOFS

PROOF OF LEMMA 1 Suppose that $-1 < \alpha < 0$. Because

$$\frac{2\alpha}{1-\alpha} < 0 < w_4,$$

it follows that

$$w_4 \frac{1-\alpha}{2\alpha} < 1 < \frac{1-\alpha}{1+\alpha};$$

thus, $w_1 > 0$ and $w_3 > 0$. The positiveness of w_5 is easily verified.

Now, suppose that $0 < \alpha < 1$. It is easy to see that $w_5 > 0$. Besides, if

$$w_4 < \frac{2\alpha}{1+\alpha},$$

then

$$w_4 \frac{1-\alpha}{2\alpha} < \frac{1-\alpha}{1+\alpha} < 1;$$

thus, $w_1 > 0$ and $w_3 > 0$.

PROOF OF LEMMA 3 Let $C = A(\theta)^\dagger$. By (8), $v > 0$ guarantees $u_0 \geq 0$ for $\pm q \in \mathbb{R}^3$ if and only if

$$\|C\, q\|_\infty \leq \frac{v}{6 k_f s\theta}.$$

In addition, observe that

$$\max_{\|q\| \leq q_{\max}} \|C\, q\|_\infty = q_{\max} \left\| c_i^r \right\|.$$

Thus, by Remark 1, thrust $v > 0$ guarantees $u_0 \geq 0$ for every torque $\|q\| \leq q_{\max}$ if and only if

$$v \geq k_f\, q_{\max} \sqrt{\frac{(s\theta)^2 \left(\ell^2 k_f^2 + 4k_t^2\right) + k_t^2 (c\theta)^2}{\ell^2 k_f^2 k_t^2 (s\theta)^2 + k_t^4 (c\theta)^2}}.$$

PROOF OF LEMMA 4 To show the existence of $v > 0$, it is enough to prove that $n_i > 0$ for every $i = 1, \ldots, 5$. Suppose $j = 1$ (failure in Rotor 1). After some calculations, it can be shown that

$$N = \frac{1}{s\theta} \begin{bmatrix} \frac{\left((s\theta)^2 \ell^2 k_f^2 + k_t^2 (c\theta)^2\right)\left(s\theta\sqrt{3}\ell k_f - k_t c\theta\right)^2}{3k_f\left(c(2\theta)\left(k_t^2 - \ell^2 k_f^2\right) + \ell^2 k_f^2 + k_t^2\right)^2} \\ \frac{1}{4k_f} \\ \frac{2k_t^2 (c\theta)^2}{3k_f\left(c(2\theta)\left(k_t^2 - \ell^2 k_f^2\right) + \ell^2 k_f^2 + k_t^2\right)} \\ \frac{1}{4k_f} \\ \frac{\left((s\theta)^2 \ell^2 k_f^2 + k_t^2 (c\theta)^2\right)\left(s\theta\sqrt{3}\ell k_f + k_t c\theta\right)^2}{3k_f\left(c(2\theta)\left(k_t^2 - \ell^2 k_f^2\right) + \ell^2 k_f^2 + k_t^2\right)^2} \end{bmatrix}.$$

Observe that $n_i \geq 0$ for every $i = 1, \ldots, 5$. In addition, if $|\tan\theta| \neq \frac{k_t}{\sqrt{3}\ell k_f}$, then $n_2 \neq 0$ and $n_5 \neq 0$. Furthermore, $n_4 = 0$ if and only if $\theta = \pi/2$.

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