Converging-Input Convergent-State and Related Properties of Time-Varying Impulsive Systems

José Luis Mancilla-Aguilar and Hernan Haimovich

***Abstract*—Very recently, it has been shown that the standard notion of stability for impulsive systems, whereby the state is ensured to approach the equilibrium only as continuous time elapses, is too weak to allow for any meaningful type of robustness in a time-varying impulsive system setting. By strengthening the notion of stability so that convergence to the equilibrium occurs not only as time elapses but also as the number of jumps increases, some facts that are well-established for time-invariant nonimpulsive systems can be recovered for impulsive systems. In this context, our contribution is to provide novel results consisting in rather mild conditions under which stability under zero input implies stability under inputs that converge to zero in some appropriate sense.**

***Index Terms*—Stability of nonlinear systems, time-varying systems, hybrid systems.**

I. Introduction

IMPULSIVE systems are dynamic systems whose state evolves continuously most of the time but may have discontinuities (called jumps or state resets) at isolated time instants [1]. The continuous evolution of an impulsive system is governed by ordinary differential equations (defined by the flow map), whereas jumps obey a static but possibly time-varying law (given by the jump map). The time instants when jumps occur are part of the system definition [2] and therefore an impulsive system is inherently time-varying even if both the flow and jump maps are time-invariant.

As opposed to the case of a nonimpulsive system, asymptotic stability in an impulsive system may be of two forms: weak or strong [3]–[8]. An equilibrium point of an impulsive system is weakly asymptotically stable when the state can be ensured to approach the equilibrium as continuous time elapses. Strong asymptotic stability requires convergence to be ensured not only as continuous time elapses but also as the number of jumps increases. The weak form of stability is the notion usually considered for impulsive systems, whereas the strong form is standard in hybrid systems [9]. When the number of jumps that occur in any time interval can be bounded in relation to the interval's length (the UIB condition, see [3]), then the weak and strong forms of stability become equivalent [8, Proposition 2.3].

External inputs such as control inputs or external disturbances may affect both the continuous and the jump behavior. For this reason, suitable notions of stability in the presence of inputs, such as input-to-state stability (ISS, [10], [11]) and integral-ISS (iISS, [12], [13]), must account for not only the continuous evolution of the input but also the instantaneous input values at jump instants [2]. In addition, the convergence ensured by these properties under zero input can be weak or strong, giving rise to the weak/strong versions of ISS [8] or iISS [3], [4], [6].

Under suitable conditions, the bounded solutions of time-invariant (nonimpulsive) continuous-time systems under inputs with (i) magnitude converging to zero or (ii) finite energy, converge to zero when the system is globally uniformly asymptotically stable under zero input. The former property is called converging-input convergent state (CICS) [14], [15] and is implied by ISS, whereas the latter is called bounded-energy input/convergent state (BEICS) [16], [17] and implied by iISS provided input energy is measured in correspondence with the iISS gain. The stability under zero input can hence be said to be robust with respect to the inclusion of inputs. These robustness results were extended to time-varying (nonimpulsive) systems in [18].

For impulsive systems, by contrast, we have recently shown that the usual weak form of the stability concepts is too weak to allow for any meaningful type of robustness under the inclusion of inputs [5]. Strengthening stability by accounting for the number of jumps in addition to elapsed time in the decay bound, as explained above, allows to recover for impulsive systems some facts that are well-established for time-invariant nonimpulsive systems, e.g., the facts that global uniform asymptotic stability under zero input (0-GUAS) and uniform bounded-energy input/bounded state (UBEBS) are jointly equivalent to iISS [6], [19], [20] and that ISS implies iISS [4], [12], [21].

The stability of impulsive systems under perturbations was previously addressed in [22]–[24]. These works impose conditions on the impulse times that imply the UIB condition, hence

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making the weak and strong versions of stability equivalent, and assume the jumps maps satisfy a Lipschitz condition.

In this context, the current main contribution is to provide rather mild conditions under which stability under zero input ensures robustness of stability under inputs whose magnitude or energy converge to zero, thus generalizing many existing results. Interesting features of our results are: i) no constraints on the impulse-time sequences are imposed, ii) no Lipschitz condition on the jump maps is assumed, and iii) Lyapunov functions are avoided, because their existence cannot be ensured under our mild assumptions.

II. Preliminaries

A. Notation and Preliminary Definitions

The reals, nonnegative reals, naturals and nonnegative integers are denoted $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{N}$ and $\mathbb{N}_0$, respectively. For $\xi \in \mathbb{R}^n$, $|\xi|$ denotes its Euclidean norm. L^1_{loc} denotes the set of all the Lebesgue measurable functions $v : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that $|v|$ is integrable on each finite interval $I \subset \mathbb{R}_{\geq 0}$. We write $\rho \in \mathcal{K}$ if $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous, strictly increasing, and $\rho(0) = 0$. We write $\rho \in \mathcal{K}_\infty$ if $\rho \in \mathcal{K}$ and ρ is unbounded. We write $\beta \in \mathcal{KL}$ if $\beta : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfies $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each $t \geq 0$ and $\beta(r, \cdot)$ is strictly decreasing to zero for each $r \geq 0$. We abbreviate "with respect to" as "w.r.t.". For $a \in \mathbb{R}$, $\lceil a \rceil$ and $\lfloor a \rfloor$ denote the least integer not less and the greatest integer not greater, respectively, than a. For every $n \in \mathbb{N}$ and $\varepsilon \geq 0$, we define the closed ball $B^n_\varepsilon := \{x \in \mathbb{R}^n : |x| \leq \varepsilon\}$.

B. Problem Statement

Consider a time-varying impulsive system with inputs:

$$\dot{x}(t) = f(t, x(t), u(t)), \quad \text{for} \quad t \notin \sigma, \tag{1a}$$

$$x(t) = x(t^-) + g(t, x(t^-), u(t)), \quad \text{for} \quad t \in \sigma, \tag{1b}$$

where $t \geq 0$, the state variable $x(t) \in \mathbb{R}^n$, the input variable $u(t) \in \mathbb{R}^m$, f (the flow map) and g (the jump map) are functions from $\mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$, and $\sigma = \{\tau_k\}_{k=1}^\infty$, with $0 < \tau_1 < \tau_2 < \cdots$ and $\lim_{k\to\infty} \tau_k = \infty$, is the impulse-time sequence. We define for convenience $\tau_0 = 0$. By "input", we mean a Lebesgue measurable function $u : [0, \infty) \to \mathbb{R}^m$ and $\mathcal{U}$ denotes the set of all the inputs.

A solution of (1) corresponding to initial time $t_0 \geq 0$, initial state $x_0 \in \mathbb{R}^n$ and input u is a function $x : [t_0, T_x) \to \mathbb{R}^n$ such that:

a) $x(t_0) = x_0$;
b) x is locally absolutely continuous on each nonempty interval of the form $J_k = [\tau_k, \tau_{k+1}) \cap [t_0, T_x)$ and $\dot{x}(t) = f(t, x(t), u(t))$ for almost all $t \in J_k$; and
c) for all $\tau_k \in (t_0, T_x)$, the left limit $x(\tau_k^-)$ exists and is finite, and it happens that

$$x(\tau_k) = x(\tau_k^-) + g(\tau_k, x(\tau_k^-), u(\tau_k)).$$

Note that b) implies that for all $t \in [t_0, T_x)$, $x(t) = x(t^+)$, i.e., x is right-continuous at t. We use $\mathcal{T}(t_0, x_0, u)$ to denote the set of maximally defined solutions of (1) corresponding to initial time t_0, initial state x_0 and input u. A solution $x \in \mathcal{T}(t_0, x_0, u)$ is forward complete if it is defined for all $t \geq t_0$. Note that even if $t_0 \in \sigma$, any solution $x \in \mathcal{T}(t_0, x_0, u)$ begins its evolution by "flowing" and not by "jumping". This is because in item II-B) above, the time instants where jumps occur are those in $\sigma \cap (t_0, T_x)$.

We associate system (1) with the zero-input system

$$\dot{x}(t) = f_0(t, x(t)), \quad \text{for} \quad t \notin \sigma, \tag{2a}$$

$$x(t) = x(t^-) + g_0(t, x(t^-)), \quad \text{for} \quad t \in \sigma, \tag{2b}$$

where $f_0(t, \xi) \equiv f(t, \xi, 0)$ and $g_0(t, \xi) \equiv g(t, \xi, 0)$.

Assumption 1: The following are satisfied:

A1) $f(\cdot, \xi, \mu)$ is Lebesgue measurable for all $(\xi, \mu) \in \mathbb{R}^n \times \mathbb{R}^m$ and $f(t, \cdot, \cdot)$ is continuous for every $t \geq 0$.

A2) (Zero-input Lipschitzianity) $f_0(t, \xi)$ is locally Lipschitz in ξ uniformly in t in the following sense: for every $R > 0$ there exists $L_R \geq 0$ such that

$$|f_0(t, \xi) - f_0(t, \xi')| \leq L_R|\xi - \xi'| \; \forall t \geq 0, \; \forall \xi, \xi' \in B^n_R.$$

A3) (Zero-input uniform continuity) $g_0(t, \xi)$ is continuous in ξ uniformly in t in the following sense: for every $R > 0$ there exists $\omega_R \in \mathcal{K}_\infty$ such that

$$|g_0(t, \xi) - g_0(t, \xi')| \leq \omega_R(|\xi - \xi'|) \; \forall t \geq 0, \; \forall \xi, \xi' \in B^n_R.$$

Assumption 1 ensures that for each $t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$ the zero-input system (2) has a unique maximally defined (forward) solution $x{:}[t_0, t_{t_0,\xi}) \to \mathbb{R}^n$ such that $x(t_0) = x_0$. In what follows we assume that the zero-input system (2) is strongly globally uniformly asymptotically stable (0-GUAS). This statement is made precise in the following assumption. For $I \subset \mathbb{R}$ we use n_I to denote the number of impulse times τ_k lying in I, i.e.,

$$n_I := \#[I \cap \sigma]. \tag{3}$$

Assumption 2 (Strong 0-GUAS): There exists $\beta \in \mathcal{KL}$ such that every zero-input system solution $x \in \mathcal{T}(t_0, x_0, 0)$ with $t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$ satisfies, for all $t \in [t_0, T_x)$,

$$|x(t)| \leq \beta\big(|x_0|, t - t_0 + n_{(t_0,t]}\big). \tag{4}$$

The stability under zero input, as characterized in Assumption 2, is called *strong* because the bound given in (4) decreases not only when continuous time advances but also whenever a jump occurs. This fact is evidenced by the quantity $t - t_0 + n_{(t_0,t]}$, which can be interpreted as the hybrid elapsed time that involves both the continuous $(t - t_0)$ and discrete $(n_{(t_0,t]})$ components. This type of strong stability is standard in the literature of hybrid systems (see [9]). Sufficient conditions for strong GUAS are given in our recent papers [7] and [8]. When the impulse-time sequence σ is uniformly incrementally bounded (UIB), i.e., $n_{(t_0,t]} \leq \phi(t - t_0)$ for all $t \geq t_0 \geq 0$ for some nondecreasing function $\phi{:}\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, strong 0-GUAS is equivalent to the most usual (weak) 0-GUAS property obtained by replacing $\beta(|x_0|, t - t_0 + n_{(t_0,t]})$ by $\beta(|x_0|, t - t_0)$ in (4) (see [7]).

The problem we address in this paper is the following:

Can we ensure that the stability of the zero-input system (0-GUAS) implies the stability of (1) *under inputs that converge to zero?*

Very recently, we have shown [5] that under weak 0-GUAS the answer to the above question is negative. This is due to the inherent nonrobustness of such a weak stability property in a time-varying impulsive systems context (see [5]).

C. The Convergence to Zero of an Input

In order to provide a positive answer to the above question under nonrestrictive assumptions, we need to make precise the notion of convergence to zero. To this aim, we require the following additional assumption.

Assumption 3: The functions f and g in (1) satisfy:

C1) (uniform boundedness) There exists $\gamma \in \mathcal{K}_\infty$ so that for $R > 0$ there exists $M = M(R) \geq 0$ such that $\max\{|f(t,\xi,\mu)|, |g(t,\xi,\mu)|\} \leq (1+\gamma(|\mu|)) \cdot M$ for all $t \geq 0$, all $\xi \in B_R^n$ and all $\mu \in \mathbb{R}^m$.

C2) (uniform continuity w.r.t. input at 0) For every $R, \varepsilon > 0$ there exists $\delta = \delta(R,\varepsilon) > 0$ such that for all $t \geq 0$, $\max\{|f(t,\xi,\mu) - f(t,\xi,0)|, |g(t,\xi,\mu) - g(t,\xi,0)|\} < \varepsilon$ if $\xi \in B_R^n$ and $|\mu| \leq \delta$.

Remark 1: Condition C3) is equivalent to the following mild boundedness condition:

D) f, g are bounded on $\mathbb{R}_{\geq 0} \times B_R^n \times B_R^m$ for every $R > 0$.

The implication D) $\Rightarrow$ C3) follows from [18, Lemma 2.1] since $\mathbb{R}^n$ is a separable and locally compact metric space. The converse implication is straightforward.

Let γ be the function given by C3) and let $\mathcal{U}_\gamma := \{u \in \mathcal{U} : \gamma(|u(\cdot)|) \in L^1_{loc}\}$. Conditions A1) and C3), the fact that a solution begins by flowing and not by jumping, and well-known results of the theory of ordinary differential equations (e.g., [25, Th. I.5.1]) ensure that for every $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ and $u \in \mathcal{U}_\gamma$ the set $\mathcal{T}(t_0, x_0, u)$ is nonempty, i.e., there exists at least one maximally defined solution $x : [t_0, t_x) \to \mathbb{R}^n$ of (1) verifying $x(t_0) = x_0$, and that if x is bounded on $[t_0, t_x)$ then it is forward complete. Thus, $\mathcal{U}_\gamma$ can be regarded as the set of *admissible* inputs. Given an interval $I \subset \mathbb{R}_{\geq 0}$ and an input $u \in \mathcal{U}$, u_I will denote the input which equals u on I and is zero elsewhere. If $I = (t_0, \infty)$ we just write u_{t_0}. We define the energy-like functional $\|\cdot\|_\gamma : \mathcal{U}_\gamma \to \mathbb{R}_{\geq 0} \cup \{\infty\}$

$$\|u\|_\gamma := \int_0^\infty \gamma(|u(s)|)ds + \sum_{s \in \sigma} \gamma(|u(s)|).$$

Note that for all $u \in \mathcal{U}_\gamma$ and every interval I of finite length,

$$\|u_I\|_\gamma = \int_I \gamma(|u(s)|)ds + \sum_{s \in \sigma \cap I} \gamma(|u(s)|) < \infty.$$

Remark 2: Given a strictly increasing sequence $\sigma = \{\tau_k\}_{k=1}^\infty$ with $\lim_{k\to\infty} \tau_k = \infty$ and $I \subset \mathbb{R}_{\geq 0}$ a Borel set, the quantity $|I| + n_I$, with $|I|$ the Lebesgue measure of I and n_I as in (3), defines a measure on the σ-algebra of Borel sets of $\mathbb{R}_{\geq 0}$. Under this interpretation, the quantity $\|u_I\|_\gamma$ becomes the integral of $\gamma(|u(\cdot)|)$ with respect to this measure.

Given $T > 1$, we say that the interval $I = (a,b] \subset \mathbb{R}_{\geq 0}$ belongs to the class $\mathcal{I}_T(\sigma)$ if $b - a + n_{(a,b]} \leq T$, and define

$$\|u\|_{\gamma,T} := \sup_{I \in \mathcal{I}_T(\sigma)} \|u_I\|_\gamma. \tag{5}$$

Considering $T > 1$ is justified in the fact that otherwise $\mathcal{I}_T(\sigma)$ would only contain intervals without impulse times.

Taking Remark 2 into account, then $\|u\|_{\gamma,T}$ is the supremum of the integrals of $\gamma(|u(\cdot)|)$ over all intervals $(a,b]$ whose measure is not greater than T. If we regard $\|u_I\|_{\gamma,T}$ as the energy of u in the interval I, then $\|u\|_{\gamma,T}/T$ could be regarded as maximum average power.

The following result establishes the relationship between using different values of T in the definition of $\|u\|_{\gamma,T}$. This relationship is important to justify the precise notion of convergence to zero that we will employ.

Proposition 1: Given $T_1, T_2 > 1$, there exists $k = k(T_1, T_2) > 0$ such that $\|u\|_{\gamma,T_1} \leq k\|u\|_{\gamma,T_2}$ for all $u \in \mathcal{U}_\gamma$.

Proof: We will establish the result with k defined as

$$k = \left\lceil \frac{T_1}{T_2 - 1} \right\rceil + 1.$$

■

Let $I = (a,b] \in \mathcal{I}_{T_1}(\sigma)$.

Claim 1: There exists a partition of I into $j \leq k$ subintervals $J_i \subset I$ satisfying $J_i \in I_{T_2}(\sigma)$ for $i = 1, \ldots, j$.

Proof of Claim 1: Let $c_0 := b$ and define recursively for $i = 1, \ldots, k$, $c_i := \inf\{c \leq c_{i-1} : c_{i-1} - c + n_{(c,c_{i-1}]} \leq T_2\}$. The function $\rho_{i-1}(t) = c_{i-1} - t + n_{(t,c_{i-1}]}$, $t \leq c_{i-1}$, is strictly decreasing, continuous from the right at each $t < c_{i-1}$, left-continuous for all t such that $t \notin \sigma$, $\rho_{i-1}(t^-) - \rho_i(t) = 1$ for all $t \in \sigma$ and $\rho_{i-1}(c_{i-1}) = 0$. Since $c_i = \inf\{c \leq c_{i-1} : \rho_{i-1}(c) \leq T_2\}$ and $T_2 > 1$ it follows that

$$0 < T_2 - 1 \leq c_{i-1} - c_i + n_{(c_i,c_{i-1}]} \leq T_2 \tag{6}$$

for $i = 1, \ldots, k$. Hence

$$\begin{aligned} c_0 - c_k + n_{(c_k,c_0]} &= \sum_{i=1}^k (c_{i-1} - c_i + n_{(c_i,c_{i-1}]}) \\ &\geq k(T_2 - 1) > T_1. \end{aligned}$$

Therefore, since $I = (a,b] \in \mathcal{I}_{T_1}(\sigma)$ and hence $b - a + n_{(a,b]} \leq T_1$, then $c_k < a$. Define $d := \min\{c_i : c_i > a\}$ and let $j \in \mathbb{N}$ satisfy $c_{j-1} = d$. Define the intervals $J_i := (c_i, c_{i-1}]$ for $i = 1, \ldots, j-1$ if $j > 1$, and $J_j := (a, d]$. From (6), then $J_i \in \mathcal{I}_{T_2}(\sigma)$ and from their definition, the subintervals $J_i \subset I$, for $i = 1, \ldots, j$, constitute a partition of I.

Using Claim 1, we have $\|u_I\|_\gamma = \sum_{i=1}^j \|u_{J_i}\|_\gamma$ and therefore, recalling the definition (5),

$$\|u\|_{\gamma,T_1} \leq j\|u\|_{\gamma,T_2} \leq k\|u\|_{\gamma,T_2}.$$

Definition 1: We say that $u \in \mathcal{U}_\gamma$ converges to 0, and write $u \xrightarrow{\text{pow}} 0$, if $\lim_{\tau\to\infty} \|u_\tau\|_{\gamma,T} = 0$. We define $\mathcal{U}_\gamma^0 := \{u \in \mathcal{U}_\gamma : u \xrightarrow{\text{pow}} 0\}$ as the set of all the admissible inputs that converge to 0.

An input converges to zero if its maximum average power over the interval (τ, ∞) is arbitrarily small for τ large enough. Note that Proposition 1 implies that neither the set $\mathcal{U}_\gamma^0$ nor the convergence $u \xrightarrow{\text{pow}} 0$ depend on the number T in the definition, since if $\lim_{\tau\to\infty} \|u_\tau\|_{\gamma,T} = 0$ for some $T > 1$ then $\lim_{\tau\to\infty} \|u_\tau\|_{\gamma,T} = 0$ for all $T > 1$.

III. Main Results

Theorem 1 is the main result of the paper.

Theorem 1: Let Assumptions 1, 2 and 3 hold and consider γ from C3) in Assumption 3. Then,

a) For every $\varepsilon > 0$ and $T > 1$ there exists $\delta = \delta(\varepsilon, T) > 0$ so that if x is a solution of (1) corresponding to $u \in \mathcal{U}_\gamma$ such that $\|u\|_{\gamma,T} \le \delta$, and $|x(t_0)| \le \delta$ for some $t_0 \ge 0$, then x is forward complete and $|x(t)| \le \varepsilon$ for all $t \ge t_0$.

b) If x is a solution of (1) corresponding to some $u \in \mathcal{U}_\gamma^0$, then the following are equivalent:

   i) $\lim_{t\to\infty} x(t) = 0$,

   ii) x is bounded.

Theorem 1a) shows that solutions of (1) that begin sufficiently close to the origin and correspond to inputs with sufficiently small power are forward complete and remain close to the origin; in other words, this shows that system (1) is totally stable as defined in [26]. Theorem 1b) answers the question posed towards the end of Section II-B. For proving Theorem 1 we require Lemma 1.

Given $L \ge 0$ and $\omega \in \mathcal{K}_\infty$, we recursively define the functions $h_j : \mathbb{R}^2_{\ge 0} \to \mathbb{R}_{\ge 0}$ as follows

$$h_0(p,t) = pe^{Lt}, \quad \text{and, for } j \ge 1, \tag{7}$$

$$h_j(p,t) = h_{j-1}(p,t) + \sup_{0\le s\le t}\Big[\omega(h_{j-1}(p,s))e^{L(t-s)}\Big]. \tag{8}$$

Lemma 1: Let Assumptions 1, 2 and 3 hold and consider β from Assumption 2 and γ from C3) in Assumption 3. For every $R > 0$ and $\eta > 0$, there exist $L = L(R) > 0$, $\kappa = \kappa(R,\eta) > 0$ and $\omega \in \mathcal{K}_\infty$ which only depends on R, such that if h_j are the functions defined via (7)–(8) and x is a solution of (1) corresponding to $u \in \mathcal{U}_\gamma$ such that $x(s) \in B_R^n$ for all $s \in [t_0, t_0+T)$, then for all $t \in [t_0, t_0+T]$, defining $j = n_{(t_0,t]}$ it follows that

$$|x(t)| \le \beta(|x(t_0)|, t - t_0 + j) + h_j\big((t-t_0+j)\eta + \kappa\|u_{(t_0,t]}\|_\gamma,\ t-t_0\big).$$

Proof: Let $R > 0$. Define $R^* = \beta(R,0)$ and consider L_{R^*} and ω_{R^*}, with L_{R^*} and ω_{R^*} as in, respectively, A1) and A1) with R^* instead of R. Let $\eta > 0$. The proof of the following result can be obtained *mutatis mutandis* from that of the claim in the proof of [18, Lemma 3.3]. ■

Claim 2: There exists $\kappa = \kappa(R^*, \eta) > 0$ such that for all $s \ge 0$, $\xi \in B_{R^*}^n$ and $\mu \in \mathbb{R}^m$,

$$|f(s,\xi,\mu) - f(s,\xi,0)| \le \eta + \kappa\gamma(|\mu|), \text{ and} \tag{9}$$

$$|g(s,\xi,\mu) - g(s,\xi,0)| \le \eta + \kappa\gamma(|\mu|). \tag{10}$$

Let x be a solution of (1) corresponding to $u \in \mathcal{U}_\gamma$ such that $x(s) \in B_R^n$ for all $s \in [t_0, t_0+T)$. Let z be the unique solution of (2) satisfying $z(t_0) = x(t_0)$. Note that $|z(s)| \le \beta(|x(t_0)|, 0) \le R^*$ for all $s \ge t_0$. So, both $x(s)$ and $z(s)$ belong to $B_{R^*}^n$ for all $s \in [t_0, t_0+T)$. Fix $t \in [t_0, t_0+T]$ and let τ satisfy $t_0 \le \tau \le t$. We have

$$x(\tau) - z(\tau) = \int_{t_0}^{\tau} [f(s,x(s),u(s)) - f(s,z(s),0)]ds + \sum_{s\in(t_0,\tau]\cap\sigma} [g(s,x(s^-),u(s)) - g(s,z(s^-),0)]$$

Adding and subtracting $f(s,x(s),0)$ and $g(s,x(s^-),0)$ within the respective square brackets, applying norms and the triangle inequality, it follows that

$$\begin{aligned}|x(\tau) - z(\tau)| \le &\int_{t_0}^{\tau} \big|f(s,x(s),u(s)) - f(s,x(s),0)\big|ds \\ &+ \int_{t_0}^{\tau} \big|f(s,x(s),0) - f(s,z(s),0)\big|ds \\ &+ \sum_{s\in(t_0,\tau]\cap\sigma} \big|g(s,x(s^-),u(s)) - g(s,x(s^-),0)\big| \\ &+ \sum_{s\in(t_0,\tau]\cap\sigma} \big|g(s,x(s^-),0) - g(s,z(s^-),0)\big| \end{aligned} \tag{11}$$

Since both $x(s) \in B_{R^*}^n$ and $z(s) \in B_{R^*}^n$ for all $s \in [t_0, t_0+T)$, employing A2), A3) and Claim 2, from (11) we arrive at

$$\begin{aligned}&|x(\tau) - z(\tau)| \\ &\le \int_{t_0}^{\tau} [\eta + \kappa\gamma(|u(s)|)]ds + L_{R^*}\int_{t_0}^{\tau} |x(s) - z(s)|ds \\ &\quad + \sum_{s\in(t_0,\tau]\cap\sigma} \big[\eta + \kappa\gamma(|u(s)|) + \omega_{R^*}\big(|x(s^-) - z(s^-)|\big)\big] \\ &\le \eta\cdot(t - t_0 + n_{(t_0,t]}) + \kappa\|u_{(t_0,t]}\|_\gamma \\ &\quad + L_{R^*}\int_{t_0}^{\tau} |x(s) - z(s)|ds + \sum_{s\in(t_0,\tau]\cap\sigma} \omega_{R^*}\big(|x(s^-) - z(s^-)|\big),\end{aligned}$$

where we have employed the fact that $\tau - t_0 + n_{(t_0,\tau]} \le t - t_0 + n_{(t_0,t]}$. Note that the latter holds even for $\tau = t_0 + T$, since $x(s^-)$ and $z(s^-)$ belong to $B_{R^*}^n$ for all $s \in [t_0, t_0+T]$. The result follows by application of Lemma V, given in the Appendix, to the function $y : [t_0, t_0+T] \to \mathbb{R}_{\ge 0}$ defined via $y(s) = |x(s) - z(s)|$, setting $p = \eta\cdot(t - t_0 + n_{(t_0,t]}) + \kappa\|u_{(t_0,t]}\|_\gamma$, $L = L_{R^*}$ and $\omega = \omega_{R^*}$, and taking into account that $|x(t)| \le |z(t)| + |x(t) - z(t)| \le \beta(|x(t_0)|, t - t_0 + n_{(t_0,t]}) + |x(t) - z(t)|$.

Proof of Theorem 1a): For the sake of simplicity, we will write $\|\cdot\|_T$ instead of $\|\cdot\|_{\gamma,T}$.

Let $\varepsilon > 0$ and $T > 1$. Let $\beta \in \mathcal{KL}$ the function coming from Assumption 2. Pick $\delta^* > 0$ and $T^* > 1$ such that $\beta(\delta^*, 0) < \varepsilon/2$ and $\beta(\delta^*, T^*) < \delta^*/2$. Let $L > 0$ and $\omega \in \mathcal{K}_\infty$ be the constant and the function corresponding to $R = \varepsilon$ in Lemma 1. Let $j^* = \lfloor T^* \rfloor + 1$.

Consider the functions h_j defined in (7)–(8). These functions are continuous, nondecreasing in each variable, $h_j \le h_k$ for all $j \le k$, and $h_j(0, r) = 0$ for all $r \ge 0$ and all j. Pick $\lambda > 0$ so that $h_{j^*}(\lambda, T^*+1) \le \delta^*/2$. For such a λ we also have that for all $0 \le p \le \lambda$, $0 \le r \le T^*+1$ and $0 \le j \le j^*$,

$$h_j(p,r) \le h_{j^*}(p, T^*+1) \le \frac{\delta^*}{2}. \tag{12}$$

From Proposition 1, there exists $c > 0$ such that $\|u\|_{T^*} \le c\|u\|_T$ for all $u \in \mathcal{U}_\gamma$. Let $\eta = \frac{0.5\lambda}{T^*+1}$ and $\kappa = \kappa(\varepsilon,\eta)$ from Lemma 1 and define $\hat{\delta} = \dfrac{\lambda}{2c\kappa}$.

Let $x \in \mathcal{T}(t_0, x_0, u)$ with $t_0 \ge 0$, $|x_0| \le \delta^*$ and $u \in \mathcal{U}_\gamma$ such that $\|u\|_T \le \hat{\delta}$. Let $[t_0, t_x)$ be the interval of definition of x. Define $t_1 = \sup\{t \ge t_0 : t - t_0 + n_{(t_0,t]} \le T^*\}$. We have that $T^* \le t_1 - t_0 + n_{(t_0,t_1]} \le T^* + 1$, hence $n_{(t_0,t_1]} \le T^*+1$ and since $n_{(t_0,t_1]} \in \mathbb{N}_0$, then $n_{(t_0,t_1]} \le j^*$. Thus, $n_{(t_0,t]} \le j^*$ for all $t_0 \le t \le t_1$.

Claim 3: $t_x > t_1$ and $x(t) \in B_\varepsilon^n$ for all $t \in [t_0, t_1]$.

Proof of Claim 3: Since $x(t_0) = x_0$ satisfies $|x_0| \leq \delta^*$, x is right continuous and $\delta^* < \varepsilon/2$, it follows that $x(s) \in B^n_\varepsilon$ for all s in some interval $[t_0, t_0']$ with $t_0' > t_0$. Let

$$\tau^* = \sup\{\tau \in [t_0, t_x) : x(s) \in B^n_\varepsilon \ \forall s \in [t_0, \tau]\},$$

then $\tau^* > t_0$. Suppose that $\tau^* \leq t_1$. Then, for every $t_0 \leq t < \tau^*$, $x(t)$ belongs to B^n_ε. In addition $t - t_0 + n_{(t_0,t]} \leq t_1 - t_0 + n_{(t_0,t_1]} \leq T^* + 1$, which implies that $t - t_0 \leq T^* + 1$ and $j = n_{(t_0,t]} \leq j^*$. By applying Lemma 1 and using the definitions of $\hat{\delta}$, η and $\|u\|_T$, and (12), it follows that

$$\begin{aligned}|x(\tau^*)| &\leq \beta(|x(t_0)|, \tau^* - t_0 + j)\\ &\quad + h_j((\tau^* - t_0 + j)\eta + \kappa\|u_{(t_0,\tau^*]}\|, \tau^* - t_0)\\ &\leq \frac{\varepsilon}{2} + h_{j^*}(\lambda, T^* + 1) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} < \varepsilon. \qquad (13)\end{aligned}$$

This shows that $x(t)$ exists for all t in some interval $[\tau^*, t']$ with $t' > \tau^*$ and that $x(t) \in B^n_\varepsilon$ on this interval, contradicting the definition of τ^*. In consequence, $\tau^* > t_1$ and the claim is proved.

Taking into account Claim 3, and reasoning as above, then for every $t \in [t_0, t_1]$ we have that $|x(t)| \leq \varepsilon$ and

$$\begin{aligned}|x(t_1)| &\leq \beta(|x(t_0)|, t_1 - t_0 + j)\\ &\quad + h_j((t_1 - t_0 + j)\eta + \kappa\|u_{(t_0,t_1]}\|, t_1 - t_0)\\ &\leq \frac{\delta^*}{2} + h_{j^*}(\lambda, T^* + 1) \leq \frac{\delta^*}{2} \leq \frac{\delta^*}{2} = \delta^*.\end{aligned}$$

since $t_1 - t_0 + n_{(t_0,t_1]} \geq T^*$. Define the strictly increasing sequence $\{t_k\}_{k=0}^\infty$ as follows

$$t_{k+1} = \sup\{t \geq t_k : t - t_k + n_{(t_k,t]} \leq T^*\}.$$

Repeating the reasoning performed above in a recursive manner we obtain that for all $k \in \mathbb{N}_0$ and all $t \in [t_k, t_{k+1}]$, $x(t)$ is defined, $|x(t)| \leq \varepsilon$ and $|x(t_{k+1})| \leq \delta^*$.

Next, we claim that $t_k \to \infty$. Suppose that the claim is not true. Since t_k is increasing, then $t_k \to \bar{t}$, with $\bar{t} > t_0$. Then $\bar{t} - t_0 + n_{(t_0,\bar{t}]} \geq t_k - t_0 + n_{(t_0,t_k]} = \sum_{j=1}^k [t_j - t_{j-1} + n_{(t_{j-1},t_j]}] \geq kT^*$, for all $k \geq 1$, which is absurd. Since $t_k \to \infty$, then $x(t)$ is defined for all $t \geq t_0$ and therefore x is forward complete, and $|x(t)| \leq \varepsilon$ for all $t \geq t_0$. Theorem 1I) follows by taking $\delta = \min\{\delta^*, \hat{\delta}\}$.

Proof of Theorem 1b): bi) $\Rightarrow$ bii). Since x is a solution, then x is right-continuous and the left-limit exists at each discontinuity instant. This means that x is bounded in every bounded set. Since, in addition, $\lim_{t\to\infty} x(t) = 0$, then x must remain bounded also as $t \to \infty$. Therefore, item bii) must hold.

bii) $\Rightarrow$ bi). For the sake of simplicity we drop the subscript γ in $\|\cdot\|_{\gamma,T}$. Pick any $\hat{T} > 1$. Let $\varepsilon > 0$ and let $\delta = \delta(\varepsilon, \hat{T})$ be as in the thesis of Theorem 1I). Since x is bounded, it is defined for all $t \geq t_0$ and $\Omega(x) \neq \emptyset$ (recall that the ω-limit set $\Omega(x)$ of x is the set of points $\xi \in \mathbb{R}^n$ such that there exists a strictly increasing and unbounded sequence of positive times $\{t_k\}$ such that $|x(t_k) - \xi| \to 0$ as $k \to \infty$). Let $\zeta \in \Omega(x)$ and let $R^* = |\zeta| + 1$. Define $R = \beta(R^*, 0) + 1$, where $\beta \in \mathcal{KL}$ is the function coming from Assumption 2. Let $T^* > 1$ be such that $\beta(R^*, T^*) \leq \delta/2$. Define $j^* = \lfloor T^* \rfloor + 1$.

Consider the functions h_j defined via (7)–(8) with L and ω as in Lemma 1. By using the same arguments as in the proof of Theorem 1a), there exists $\lambda > 0$ so that for all $0 \leq p \leq \lambda$, $0 \leq r \leq T^* + 1$ and $0 \leq j \leq j^*$, $h_j(p, r) \leq \min\{\frac{\delta}{2}, \frac{1}{2}\}$. Let $\eta = \frac{0.5\lambda}{T^*+1}$ and $\kappa = \kappa(R, \eta)$ given by Lemma 1. Since ζ in $\Omega(x)$, there exists a sequence $\{t_j\}$, with $t_j \to \infty$, such that $x(t_j) \to \zeta$. Taking into account that $u \in \mathcal{U}^0_\gamma$, we can pick j_0 large enough so that $|x(t_{j_0})| < R^*$, $\kappa\|u_{t_{j_0}}\|_{T^*+1} < \lambda/2$ and $\|u_{t_{j_0}}\|_{\hat{T}} < \delta$.

Define $\tau = \sup\{t \geq t_{j_0} : |x(s)| \leq R$ for all $s \in [t_{j_0}, t]\}$. Since $|x(t_{j_0})| < R$ and x is right-continuous, $\tau > t_{j_0}$. Suppose that $\tau - t_{j_0} + n_{(t_{j_0},\tau]} \leq T^* + 1$. Then $|x(t)| \leq R$ for all $t \in [t_{j_0}, \tau)$. Let $j = n_{(t_{j_0},\tau]}$. Then $j \leq j^*$. By applying Lemma 1, it follows that

$$\begin{aligned}|x(\tau)| &\leq \beta(|x(t_{j_0})|, \tau - t_{j_0} + j)\\ &\quad + h_j((\tau - t_{j_0} + j)\eta + \kappa\|u_{(t_{j_0},\tau]}\|, \tau - t_0)\\ &\leq \beta(R^*, 0) + h_{j^*}(\lambda, T^* + 1) < R.\end{aligned}$$

The right-continuity of x implies that $|x(t)| < R$ for $t \in [\tau, \tau + \nu)$, with $\nu > 0$ small enough, contradicting the definition of τ. So, $\tau - t_{j_0} + n_{(t_{j_0},\tau]} > T^* + 1$. Let $t^* = \sup\{t \geq t_{j_0} : t - t_{j_0} + n_{(t_{j_0},t]} \leq T^*\}$. Then $T^* \leq t^* - t_{j_0} + n_{(t_{j_0},t^*]} < T^* + 1$. Note that $t^* < \tau$. Taking $j = n_{(t_{j_0},t^*]}$ and reasoning as above, it follows that

$$\begin{aligned}|x(t^*)| &\leq \beta(|x(t_{j_0})|, t^* - t_{j_0} + j)\\ &\quad + h_j((t^* - t_{j_0} + j)\eta + \kappa\|u_{(t_{j_0},t^*]}\|, t^* - t_{j_0})\\ &\leq \beta(R^*, T^*) + h_{j^*}(\lambda, T^* + 1) \leq \delta.\end{aligned}$$

From the latter, the facts that $\|u_{t^*}\|_{\hat{T}} \leq \|u_{t_{j_0}}\|_{\hat{T}} \leq \delta$ and that the restriction of x to $[t^*, \infty)$ belongs to $\mathcal{T}(t^*, x(t^*), u_{t^*})$, and the selected δ, we have that $|x(t)| \leq \varepsilon$ for all $t \geq t^*$. We have established that $x \to 0$.

IV. CICS and BEICS

For an input $u \in \mathcal{U}$ we consider the supremum norm

$$\|u\|_\infty := \max\left\{\text{ess sup}_{t\geq 0}|u(t)|, \sup_{t\in\sigma}|u(t)|\right\}, \qquad (14)$$

Note that every $u \in \mathcal{U}$ such that $\|u\|_\infty < \infty$ belongs to $\mathcal{U}_\gamma$ for every $\gamma \in \mathcal{K}_\infty$.

Definition 2: Let $\gamma \in \mathcal{K}_\infty$. The system (1) is said to be

a) converging-input convergent-state (CICS) if every bounded solution $x \in \mathcal{T}(t_0, x_0, u)$, with $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$, and $u \in \mathcal{U}$ such that $\|u\|_\infty < \infty$ and $\|u_t\|_\infty \to 0$ as $t \to \infty$, satisfies $x \to 0$;

b) γ-bounded-energy-input convergent-state (BEICS) if every bounded solution $x \in \mathcal{T}(t_0, x_0, u)$, with $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$, and $u \in \mathcal{U}_\gamma$ such that $\|u\|_\gamma < \infty$ satisfies $x \to 0$.

The following results are corollaries of our main result.

Corollary 1: Let Assumptions 1, 2 and 3 hold and let γ be given by C3) in Assumption 3. Then (1) is γ-BEICS.

Proof: Let $x \in \mathcal{T}(t_0, x_0, u)$, with $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$, and $u \in \mathcal{U}_\gamma$ such that $\|u\|_\gamma < \infty$, be bounded. The fact that $\|u\|_\gamma < \infty$ implies that $\|u_t\|_\gamma \to 0$ as $t \to \infty$ and then that $u \xrightarrow{\text{pow}} 0$. By applying Theorem 1b) it then follows that $x \to 0$. ■

Corollary 2: Let Assumptions 1, 2 and 3 hold. Then, (1) is CICS.

Proof: Let $x \in \mathcal{T}(t_0, x_0, u)$, with $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$, and $u \in \mathcal{U}$ such that $\|u\|_\infty < \infty$ and $\lim_{t\to\infty} \|u_t\|_\infty = 0$, be bounded. Let $\gamma \in \mathcal{K}_\infty$ be as in C3) of Assumption 3. Then $u \in \mathcal{U}_\gamma$. Pick any $T > 1$. Let $\varepsilon > 0$. Since $\lim_{t\to\infty} \|u_t\|_\infty = 0$, there exists $t_\varepsilon > 0$ such that $\|u_t\|_\infty < \gamma^{-1}(\frac{\varepsilon}{2T})$ for all $t \geq t_\varepsilon$. Then, for any $t \geq t_\varepsilon$ and any interval $I = (a, b] \subset [t, \infty)$ such that $b - a + n_{(a,b]} \leq T$ we have that

$$\begin{aligned} \|u_{(a,b]}\|_\gamma &= \int_a^b \gamma(|u(s)|)\, ds + \sum_{s\in\sigma\cap(a,b]} \gamma(|u(s)|) \\ &\leq (b - a + n_{(a,b]})\frac{\varepsilon}{2T} < \varepsilon. \end{aligned}$$

Therefore, $\|u_t\|_{\gamma,T} \leq \varepsilon$ for all $t \geq t_\varepsilon$. This shows that $u \in \mathcal{U}_\gamma^0$. Applying Theorem 1b) we obtain that $x \to 0$. ■

V. Conclusion

We have provided, for the first time in an impulsive time-varying system setting, conditions that ensure that stability under zero input is inherited under inputs that converge to zero in some appropriate sense. These conditions constitute rather mild boundedness and continuity requirements on the flow and jump maps. The given novel results have the following salient features: the jump map is not required to satisfy any kind of Lipschitz continuity property and solutions under nonzero inputs are not necessarily unique.

Appendix

The following comparison-type result can be seen as a corollary to [4, Lemma 3.2] and is required in the proof of Lemma 1.

Lemma 1: Let $0 \leq d < b$ and let $y : [d, b] \to \mathbb{R}_{\geq 0}$ be a right-continuous function having a finite number N of points of discontinuity $s_1, \ldots, s_N$ satisfying $d < s_1 < \cdots < s_N \leq b$. Let y be such that the left-limit $y(s_j^-)$ exists for all $j = 1, \ldots, N$. Let $p \in \mathbb{R}_{\geq 0}$, $\omega \in \mathcal{K}_\infty$ and $\sigma = \{s_k\}_{k=1}^N$. If y satisfies

$$y(t) \leq p + L\int_d^t y(s)ds + \sum_{s\in\sigma\cap(d,t]} \omega(y(s^-)) \qquad (15)$$

for all $t \in [d, b]$, then in the same time interval y satisfies

$$y(t) \leq h_k(p, t - d), \qquad (16)$$

where $k = n_{(d,t]} = \#[\sigma \cap (d, t]]$, and the functions $h_j:\mathbb{R}_{\geq 0} \times [0, \infty) \to \mathbb{R}_{\geq 0}$, $j = 0, 1, \ldots$, are recursively defined by (7)-(8).

Proof: Define the constant function $a : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $a(s) \equiv L$, and the constant sequence of nonnegative numbers $\{c_i\}_{i=1}^N$, $c_i \equiv 1$. Then, the assumptions of [4, Lemma 3.2] are satisfied and application of this lemma yields $y(t) \leq h_k^d(p, t)$ with the functions $h_j^d : \mathbb{R}_{\geq 0}^2 \to \mathbb{R}_{\geq 0}$ defined recursively as $h_0^d(p, t) = pe^{L(t-d)}$ and for $j \geq 1$,

$$\begin{aligned} h_j^d(p, t) &= h_{j-1}^d(p, t) + e^{L(t-d)} \sup_{d\leq s\leq t}\Big[\omega(h_{j-1}^d(p, s))e^{-L(s-d)}\Big] \\ &= h_{j-1}^d(p, t) + \sup_{d\leq s\leq t}\Big[\omega(h_{j-1}^d(p, s))e^{L(t-s)}\Big]. \end{aligned}$$

With these definitions, it follows that $h_j^d(p, t) = h_j^0(p, t-d) = h_j(p, t - d)$ for all $j \in \mathbb{N}_0$, $p \geq 0$, and $t \geq d \geq 0$. ■

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