

Supplementary material to: An arbitrary order Mixed Virtual Element formulation for coupled multi-dimensional flow problems

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Table 0.1: Dimensions for various polynomial spaces for different polynomial orders

k	2D						
	0	1	2	3	4	5	6
$\dim(\mathbb{P}_k(E))$	1	3	6	10	15	21	28
$\dim(\mathbb{P}_k(E))^2$	2	6	12	20	30	42	56
$\dim(\nabla \mathbb{P}_k(E))$	0	2	5	9	14	20	27
$\dim(\nabla \mathbb{P}_{k+1}(E))$	2	5	9	14	20	27	35
$\dim(\nabla \mathbb{P}_{k+1}(E))^\oplus$	0	1	3	6	10	15	21

k	3D						
	0	1	2	3	4	5	6
$\dim(\mathbb{P}_k(E))$	1	4	10	20	35	56	84
$\dim(\mathbb{P}_k(E))^3$	3	12	30	60	105	168	252
$\dim(\nabla \mathbb{P}_k(E))$	0	3	9	19	34	55	83
$\dim(\nabla \mathbb{P}_{k+1}(E))$	3	9	19	34	55	83	120
$\dim(\nabla \mathbb{P}_{k+1}(E))^\oplus$	0	3	11	26	60	85	132

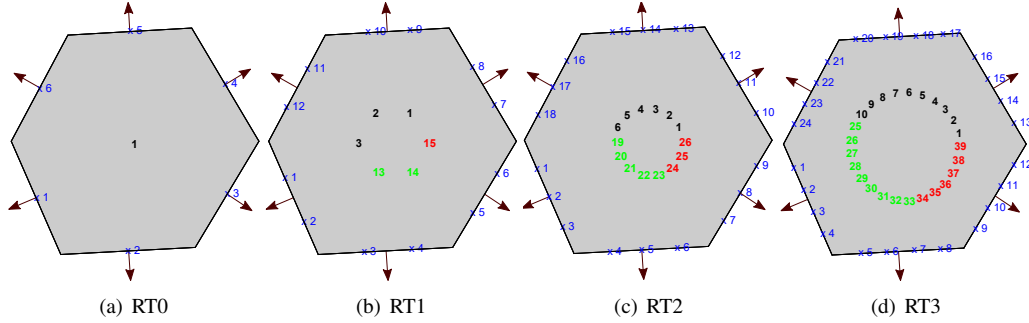


Figure 0.1: 2D DOFs for the lowest order elements. DOFs of types i, ii and iii are represented in blue, red and green respectively, while internal pressure DOFs are in black

Basis for polynomials spaces. Tables 0.3 and 0.4 list all polynomials comprising the space of gradients and its complement. Figure 0.4 is a schematic representation of Pascal's pyramid, which contains the ordering of the polynomials used in the implementation of the method. $\bar{x}$, $\bar{y}$ and $\bar{z}$ should be understood as $\frac{(x-x_0)}{d}$, $\frac{(y-y_0)}{d}$ and $\frac{(z-z_0)}{d}$, where the subscript 0 indicates the barycenter of the element a d its diameter.

Table 0.2: Number of degrees of freedom for local VEM spaces in 2D and 3D

k	DOF type	2D RTk-VEM		3D RTk-VEM		3D BDMk-VEM	
		$V_{\ell,h}^E$	$Q(E)$	$V_{\ell,h}^E$	$Q(E)$	$V_{\ell,h}^E$	$Q(E)$
0	i	n_e		n_f			
	ii	0	1	0	1	-	-
	iii	0		0			
1	i	$2n_e$		$3n_f$		$3n_f$	
	ii	2	3	3	4	0	1
	iii	1		3		3	
2	i	$3n_e$		$6n_f$		$6n_f$	
	ii	5	6	9	10	3	4
	iii	3		11		11	
3	i	$4n_e$		$10n_f$		$10n_f$	
	ii	9	10	19	20	9	10
	iii	6		26		26	
4	i	$5n_e$		$15n_f$		$15n_f$	
	ii	14	15	34	35	19	20
	iii	10		50		50	
5	i	$6n_e$		$21n_f$		$21n_f$	
	ii	20	21	55	56	34	35
	iii	15		85		85	
6	i	$7n_e$		$28n_f$		$28n_f$	
	ii	27	28	83	84	55	56
	iii	21		133		133	

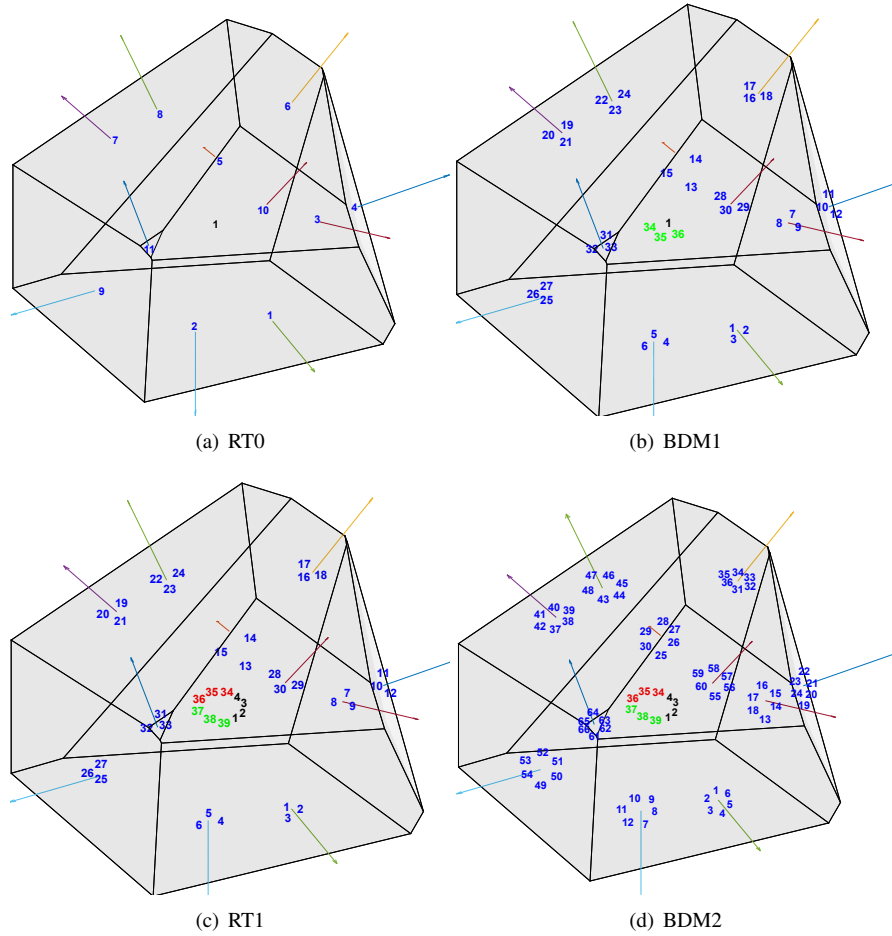


Figure 0.2: 3D DOFs for the lowest order elements. DOFs of types i, ii and iii are represented in blue, red and green respectively, while internal pressure DOFs are in black

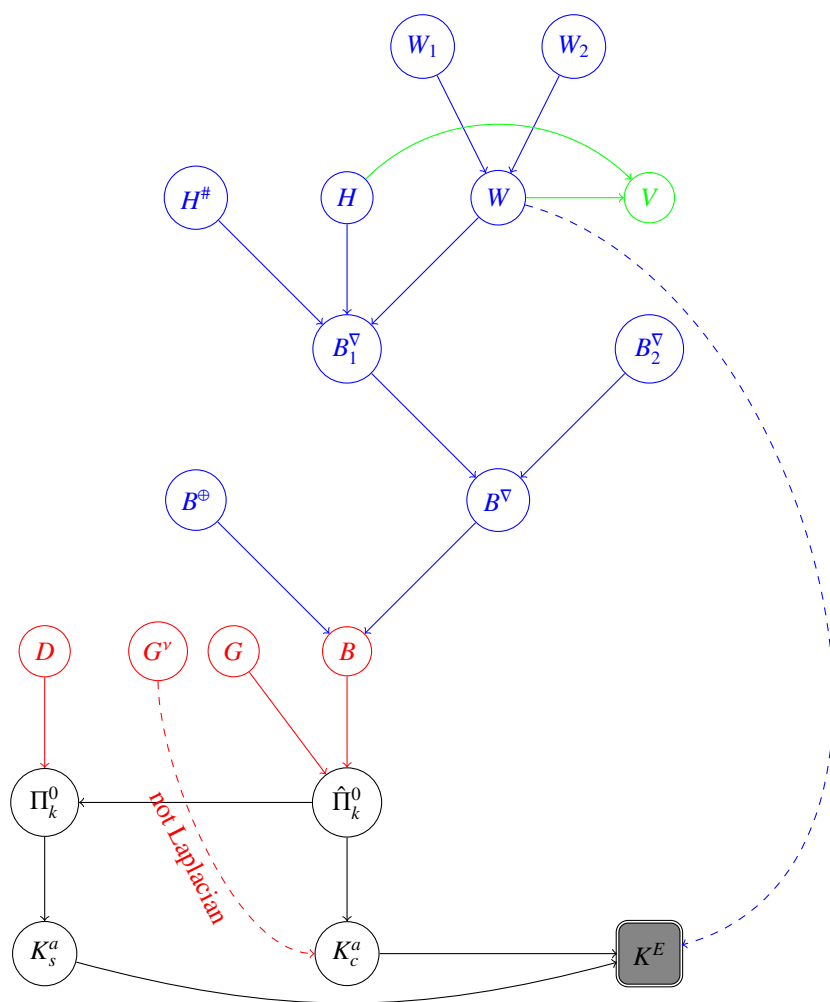


Figure 0.3: Flow chart of matrix computations

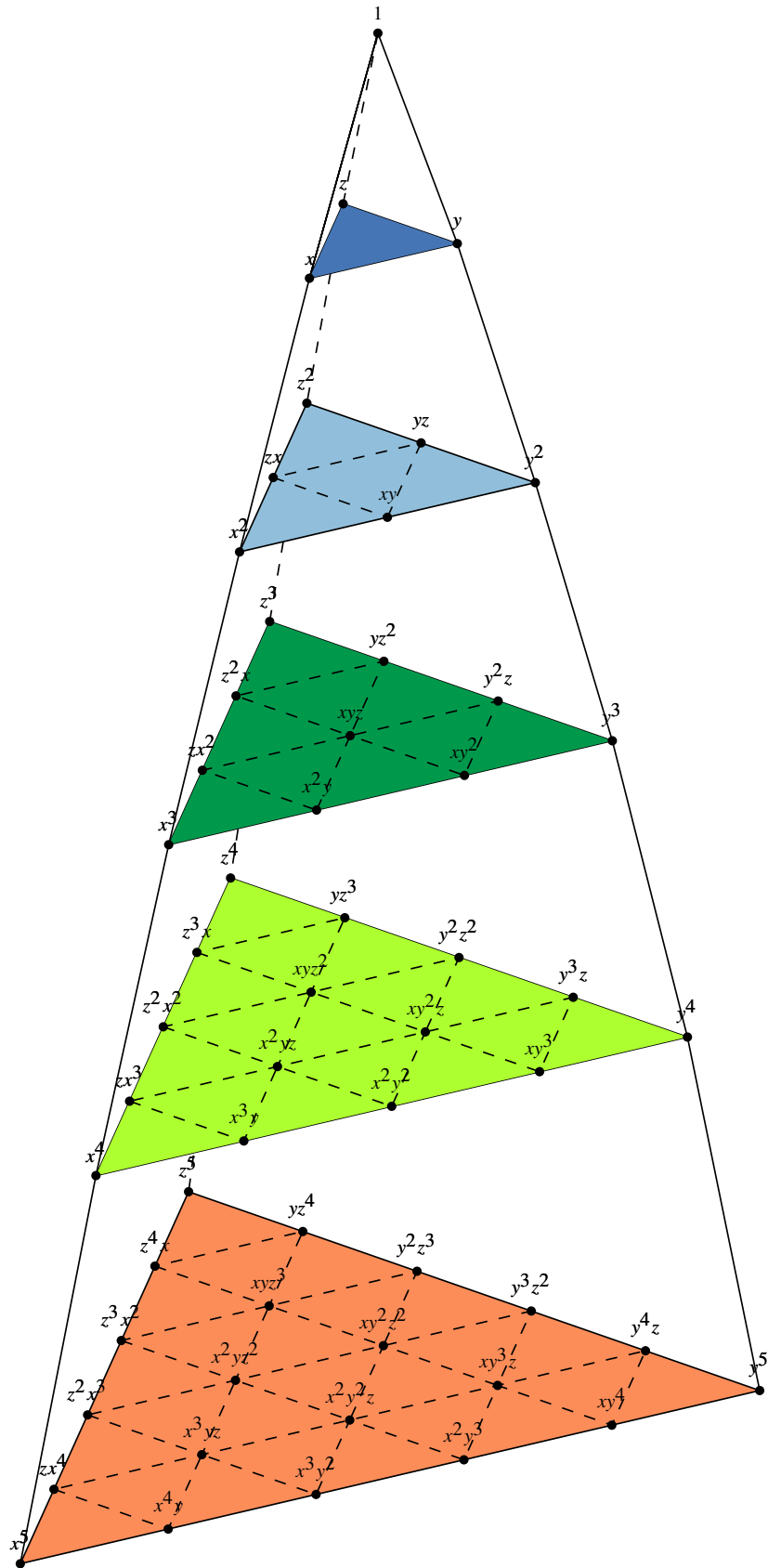


Figure 0.4: Pascal's pyramid

Table 0.3: Decomposition of spaces $\mathcal{G}_{0,1,2,3}$

k	#	$\nabla \mathbb{P}_{k+1}$	$(\nabla \mathbb{P}_{k+1})^\oplus$
0 $\frac{1}{d}$	1	$(1, 0, 0)$	
	2	$(0, 1, 0)$	
	3	$(0, 0, 1)$	
$\frac{1}{d^2}$ $\frac{1}{d}$	4	$(2x, 0, 0)$	$(y, -x, 0)$ $(z, 0, -x)$ $(0, z, -y)$
	5	$(y, x, 0)$	
	6	$(0, 2y, 0)$	
	7	$(0, z, y)$	
	8	$(0, 0, 2z)$	
	9	$(z, 0, x)$	
	10		
	11		
	12		
	13	$(3x^2, 0, 0)$	$(xy, -2x^2, 0)$ $(-2y^2, -xy, 0)$ $(yz, 0, -xy)$ $(0, yz, -2y^2)$ $(0, 2z^2, yz)$ $(xz, 0, -2x)$ $(0, xz, -xy)$ $(-2z^2, 0, xz)$
	14	$(2xy, x^2, 0)$	
	15	$(y^2, 2xy, 0)$	
	16	$(0, 3y^2, 0)$	
	17	$(0, 2yz, y^2)$	
	18	$(0, z^2, 2yz)$	
	19	$(0, 0, 3z^2)$	
	20	$(z^2, 0, 2xz)$	
	21	$(2xz, 0, x^2)$	
	22	(yz, xz, xy)	
	23		
	24		
	25		
	26		
	27		
	28		
	29		
	30		

k	#	$\nabla \mathbb{P}_{k+1}$	$(\nabla \mathbb{P}_{k+1})^\oplus$
3 $\frac{1}{d^4}$	31	$(4x^3, 0, 0)$	
	32	$(3x^2y, x^3, 0)$	
	33	$(2xy^2, 2x^2y, 0)$	
	34	$(y^3, 3xy^2, 0)$	
	35	$(0, 4y^3, 0)$	
	36	$(0, 3y^2z, y^3)$	
	37	$(0, 2yz^2, 2y^2z)$	
	38	$(0, z^3, 3yz^2)$	
	39	$(0, 0, 4z^3)$	
	40	$(z^3, 0, 3xz^2)$	
	41	$(2xz^2, 0, 2x^2z)$	
	42	$(3x^2z, x^2z, x^3)$	
	43	$(2xyz, x^2z, x^2y)$	
	44	$(y^2z, 2xyz, xy^2)$	
	45	$(yz^2, xz^2, 2xyz)$	
$\frac{1}{d^3}$	46		$(x^2y, -3x^3, 0)$
	47		$(xy^2, -x^2y, 0)$
	48		$(y^3, \frac{-1}{3}xy^2, 0)$
	49		$(y^2z, 0, -xy^2)$
	50		$(0, y^2z, -3y^3)$
	51		$(0, yz^2, -y^2z)$
	52		$(0, z^3, \frac{-1}{3}yz^2)$
	53		$(-yz^2, xz^2, 0)$
	54		$(-3z^3, 0, xz^2)$
	55		$(x^2z, 0, -3x^3)$
	56		$(0, x^2z, -x^2y)$
	57		$(-xz^2, 0, x^2z)$
	58		$(xyz, 0, -2x^2y)$
	59		$(0, xyz, -2xy^2)$
	60		$(-2yz^2, 0, xyz)$

Table 0.4: Decomposition of spaces $\mathcal{G}_{4,5}$

k	#	$\nabla\mathbb{P}_{k+1}$	$(\nabla\mathbb{P}_{k+1})^{\oplus}$
$\frac{4}{d^5}$	61	$(5x^4, 0, 0)$	
	62	$(4x^3y, x^4, 0)$	
	63	$(3x^2y^2, 2x^3y, 0)$	
	64	$(2xy^3, 3x^2y^2, 0)$	
	65	$(y^4, 4xy^3, 0)$	
	66	$(0, 5y^4, 0)$	
	67	$(0, 4y^3z, y^4)$	
	68	$(0, 3y^2z^2, 2y^3z)$	
	69	$(0, 2yz^3, 3y^2z^2)$	
	70	$(0, z^4, 4yz^3)$	
	71	$(0, 0, 5z^4)$	
	72	$(z^4, 0, 4xz^3)$	
	73	$(2xz^3, 0, 3x^2z^2)$	
	74	$(3x^2z^2, 0, 2x^3z)$	
	75	$(4x^3z, 0, x^4)$	
	76	$(3x^2yz, x^3z, x^3y)$	
	77	$(2xy^2z, 2x^2yz, x^2y^2)$	
	78	$(y^3z, 3xy^2z, xy^3)$	
	79	$(y^2z^2, 2xy^2z, 2xy^2)$	
	80	$(yz^3, xz^3, 3xyz^2)$	
	81	$(2xyz^2, x^2z^2, 2x^2yz)$	
$\frac{1}{d^4}$	82		$(-3yz^3, 0, xyz^2)$
	83		$(-2yz^2, xyz^2, 0)$
	84		$(xyz^2, -2x^2z^2, 0)$
	85		$(-2y^2z^2, 0, xyz^2)$
	86		$(0, xyz^2, -3xy^3)$
	87		$(xy^2z, 0, -2x^2y^2)$
	88		$(0, -2x^2z^2, -2x^2y^2)$
	89		$(0, x^2yz, x^2yz)$
	90		$(x^2yz, 0, -2x^2y^2)$
	91		$(-2x^2z^2, 0, 3x^3z)$
	92		$(0, x^3z, -3x^3y)$
	93		$(x^3z, 0, -4x^4)$
	94		$(-3xz^3, 0, 2x^2z^2)$
	95		$(-4z^4, 0, xz^3)$
	96		$(-yz^3, xz^3, 0)$
	97		$(0, 4z^4, -yz^3)$
	98		$(0, 3yz^3, -2y^2z^2)$
	99		$(0, 2y^2z^2, -3y^3z)$
	100		$(0, y^3z, -4y^4)$
	101		$(y^3z, 0, -3xy^3)$
	102		$(4y^4, -xy^3, 0)$
	103		$(3xy^3, -2x^2y^2, 0)$
	104		$(2x^2y^2, -3x^3y, 0)$
	105		$(x^3y, -4x^4, 0)$
$\frac{5}{d^6}$	106	$(6x^5, 0, 0)$	
	107	$(5x^4y, x^5, 0)$	
	108	$(4x^3y^2, 2yx^4, 0)$	
	109	$(3x^2y^3, 3x^3y^2, 0)$	
	110	$(2xy^4, 4x^2y^3, 0)$	
	111	$(y^5, 5xy^4, 0)$	
	112	$(0, 6y^5, 0)$	
	113	$(0, 5y^4z, y^5)$	
	114	$(0, 4y^3z^2, 2yz^4)$	
	115	$(0, 3y^2z^3, 3y^3z^2)$	
	116	$(0, 2yz^4, 4y^2z^3)$	
	117	$(0, z^5, 5yz^4)$	
	118	$(0, 0, 6z^5)$	
	119	$(z^5, 0, 5xz^4)$	
	120	$(2xz^4, 0, 4x^2z^3)$	
	121	$(3x^2z^3, 0, 3x^3z^2)$	
	122	$(4x^3z^2, 0, 2xz^4)$	
	123	$(5x^4z, 0, x^5)$	
	124	$(4x^3yz, x^4z, x^4y)$	
	125	$(3x^2y^2z, 2yx^3z, x^3y^2)$	
	126	$(2xy^3z, 3x^2y^2z, x^2y^3)$	
	127	$(y^4z, 4xy^3z, xy^4)$	
	128	$(y^3z^2, 3xy^2z^2, 2zxy^3)$	
	129	$(y^2z^3, 2yxz^3, 3xy^2z^2)$	
	130	$(yz^4, xz^4, 4xyz^3)$	
	131	$(2xyz^3, x^2z^3, 3x^2yz^2)$	
	132	$(3x^2yz^2, x^3z^2, 2zx^3y)$	
	133	$(2xy^2z^2, 2yxz^2, 2xz^2y^2)$	
$\frac{1}{d^5}$	134		$(0, -3x^2z^3, x^2yz^2)$
	135		$(0, x^2yz^2, -x^2y^2z)$
	136		$(x^2yz^2, -3x^3z^2, 0)$
	137		$(-4yz^4, 0, xyz^3)$
	138		$(-2y^2z^3, xyz^3, 0)$
	139		$(xyz^3, -2x^2z^3, 0)$
	140		$(-3y^2z^3, 0, xy^2z^2)$
	141		$(-3y^3z^2, xy^2z^2, 0)$
	142		$(xy^2z^2, 0, -x^2y^2z)$
	143		$(-2y^3z^2, 0, xy^3z)$
	144		$(0, xy^3z, -4xy^4)$
	145		$(xy^3z, 0, -2x^2y^3)$
	146		$(0, x^2y^2z, -3x^3y^3)$
	147		$(x^2y^2z, 0, -3x^3y^2)$
	148		$(0, -2x^3z^2, x^3yz)$
	149		$(0, x^3yz, -2x^3y^2)$
	150		$(x^3yz, 0, -4x^4y)$
	151		$(-x^3z^2, 0, 2x^4z)$
	152		$(0, x^4z, -x^4y)$
	153		$(x^4z, 0, -5x^5)$
	154		$(-x^2z^3, 0, x^3z^2)$
	155		$(-2xz^4, 0, x^2z^3)$
	156		$(-5z^5, 0, xz^4)$
	157		$(-yz^4, xz^4, 0)$
	158		$(0, 5z^5, -yz^4)$
	159		$(0, 2yz^4, -y^2z^3)$
	160		$(0, y^2z^3, -y^3z^2)$
	161		$(0, y^3z^2, -2y^4z)$
	162		$(0, y^4z, -5y^5)$
	163		$(y^4z, 0, -xy^4)$
	164		$(5y^5, -xy^4, 0)$
	165		$(2xy^4, -x^2y^3, 0)$
	166		$(x^2y^3, -x^3y^2, 0)$
	167		$(x^3y^2, -2x^4y, 0)$
	168		$(x^4y, -5x^5, 0)$